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*Research Article*

# Generalizing the Black and Scholes Equation Assuming Differentiable Noise

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This article develops probability equations for an asset value through time, assuming continuous correlated differentiable Gaussian distributed noise. Ito's (1944) stochastic integral and a generalized Novikov's (1919) theorem are used. As an example, the mathematical model is used to generalize the Black and Scholes' (1973) equation for pricing financial instruments. The article connects the Kolmogorov (1931) probability equation to arbitrage and to how financial instruments are priced, where more generally, the mathematical model based on differentiable noise may improve or be an alternative for forecasts. The article contrasts with much of the literature which assumes continuous nondifferentiable uncorrelated Gaussian distributed white noise.

**Keywords:** Black and Scholes; correlation; differentiable noise; Fokker–Planck; Kolmogorov; probability density

## 1. Introduction

Essential in the pricing of financial instruments is Black and Scholes' [1] partial differential equation, which is based on white noise and the Ito [2] stochastic integral. Differential equations with continuous nondifferentiable uncorrelated Gaussian distributed noise, also known as the Wiener process or Brownian motion, are widespread in science [3, 4]. The Kalman–Bucy filter [5, 6] in navigation is one application of white noise and the Ito [2] stochastic integral. The filter has a procedure for estimating the state of a system that satisfies a linear differential equation based on a series of noisy observations.

The noise in this article is continuous and differentiable with nonzero correlation time, which allows the Riemann–Stieltjes integration. Correlation time is the amount of time for the noise signal to repeat in a statistical sense. Differentiable noise enables forecasts in a strong sense. Forecasts using differential equations involving noise are important in many sciences, for example, finance and navigation. In finance, an arbitrage investor may in principle spot the direction of price movements of various securities and exploit market opportunities by trading costlessly and continuously through time.

This article has a general objective, that is, the development of probability differential equations for an asset value through time, based on continuous differentiable noise with correlated Gaussian distributed noise. The correlation function can be used for forecasts. This is exemplified by generalizing the Black and Scholes' [1] equation. The article furthermore develops a Kolmogorov equation from a generalized Novikov's [7] theorem for Gaussian noise. Realizations and solutions based on random draws from probability distributions are studied. The literature develops the Kolmogorov [8] equation (also known as the Fokker–Planck equation [9, 10]) from Ito's [2] interpretation of the Langevin [11, 12] equation with noise.

It is reasonable that, according to classical physics, noise is in principle differentiable. However, on practical time scales, noise may vary too much to be observed as differentiable. This may cause accuracy problems in numerical integrations since the results depend too much on the discrete time points for integration. This problem can be solved by developing a mathematical theory that does not apply differentiability but calculates expectations of integrals involving noise. This is done in the Ito [2] or Fisk–Stratonovich [13, 14] stochastic integrals. Alternatively, as in this article, one may assume that noise is differentiable, but that explicit solutions only are achievable statistically. That means that the noise is only described through joint cumulants. Differentiable noise is easier to handle due to the Riemann–Stieltjes integration. Nondifferentiable noise may be more practical for numerical calculations since noise can be drawn randomly at different points in time by a computer. Analytically, stochastic integrals based on the Fisk–Stratonovich [13, 14] stochastic integrals and not the Ito [2] stochastic integral can provide a better mathematical approach since solutions coincide with differentiable uncorrelated Gaussian noise. See Sussmann [15] for a more exhaustive discussion about differentiable and nondifferentiable noise.

Differentiable noise has been used in finance [16] and physics [17], and fractional Brownian motion introduces a correlation between noise increments [18, 19]. In finance, the correlation in fractional Brownian motion can be used to achieve statistical arbitrages [20–23]. More generally, fractional Brownian motion is used as a model for the arrivals of network packets, involving researchers in theoretical physics, hydrology, and biology [24, 25]. These so-called stochastic differential equations have been solved through Ito's [2] and Fisk–Stratonovich's [13, 14] development of stochastic integrals [13, 14]. Other types of stochastic integrals have been developed, for example, by Hu and Øksendal [22].

The path integral formalism originally developed within quantum mechanics also applies a type of stochastic integral. The method was subsequently further developed in nonequilibrium statistical mechanics and is more general than Ito's [2] stochastic integral since it can be applied to a broad range of distributions, not only Gaussian distributions [26]. Kleinert [26] suggests that non-Gaussian noise may be important, especially in finance.

For a large class of semielliptic second-order partial differential equations, a corresponding Dirichlet [27] boundary problem can be solved by the solutions of associated stochastic differential equations. The Feynman–Kac [28] equation offers a method of solving partial differential equations by simulating stochastic paths [29]. In quantum chemistry, stochastic methods are used to solve the Schrödinger equation with the Pure Diffusion Monte Carlo method [30, 31].

Section 2 presents the Langevin [11, 12] differential equation and discusses relations for correlated and uncorrelated noise, not necessarily Gaussian. Section 3 uses a generalized Novikov's [7] theorem to develop the partial differential equation for differentiable noise, with correlated or uncorrelated Gaussian distributed noise. Section 4 generalizes the Black and Scholes [1] partial differential equation. Section 5 conducts a random draw to give an equivalent and associated stochastic process. Section 6 considers limitations, opportunities, and future research. Section 7 concludes.

## 2. Differential Equation and Correlation Relations for Differentiable Noise

Let $S(t)$ be given by the first-order ordinary Langevin [11, 12] differential equation.

$$\dot{S}(t) = G_1(t, S(t)) + G_2(t, S(t))\xi(t) \tag{1}$$

where "dot" means time derivative and $S(t)$ is a real, continuous, and differentiable function with respect to time $t \geq t_0 \geq 0$. $S(t)$ may be the value of an asset through time. For simplicity, we usually write $S(t)$ for $S(t, t_0, s_0)$, where $S(t_0, t_0, s_0) = s_0$ is the initial asset value at time $t_0$. The functions $G_1(t, S(t))$ and $G_2(t, S(t))$ are externally given real functions which are continuous and differentiable with respect to $t$ and $S(t)$. $\xi(t) = \dot{\zeta}(t)$ is the noise derivative which is an externally given real, continuous function with respect to time $t$, that is, the time derivative of the noise $\zeta(t)$ which is continuous and differentiable.

In (1), $\xi(t)$ may be random only at time $t = t_0$. Riemann–Stieltjes integration is possible since $\xi(t)$ is well behaved. Equation (1) has solutions, meaning that given the initial conditions of $t_0$ and $s_0$, the future behavior of $S(t)$ is completely determined. See Appendix A for examples of differentiable noise. For a definition of a stochastic variable and a stochastic process, see Appendix B. Nondifferentiable noise is practically drawn stochastically at each point in time $t$ throughout the time development. See Sussmann [15] for exhaustive mathematical precision.

The noise derivative $\xi(t)$ is assumed to have zero expectation, nonzero variance, and nonzero correlation, that is,

$$E(\xi(t)) = 0, E\left(\xi(t)\xi\left(t'\right)\right) = \frac{\sigma^2}{2\tau}\Pi\left(\left|t - t'\right|\right) \tag{2}$$

where "$E$" means expectation, $\Pi$ () is some function to be defined, $\tau$ is a correlation time, and $\sigma^2$ is the strength of the correlation. Correlation time $\tau$ is the amount of time for the noise signal to repeat in a statistical sense. In general, the correlation strength $\sigma^2$ may depend on time $t$ but is set time-independent unless otherwise specified.

We define

$$\Gamma(t, t_0) \equiv \int_{t_0}^{t} E(\xi(t)\xi(\nu))d\nu, \widetilde{\Gamma}(t, t_0) \equiv \int_{t_0}^{t} E(\xi(t_0)\xi(\nu))d\nu \tag{3}$$

where $\Gamma(t, t_0)$ expresses the integral of the expected value of the product of the noise derivative $\xi(t)$ at the actual time $t$ and the noise derivative $\xi(\nu)$ at time $\nu$, integrated from $t_0$ to $t$ of the variable $\nu$, and $\widetilde{\Gamma}(t, t_0)$ expresses the integral of the expected value of the product of the noise derivative $\xi(t_0)$ at the initial time $t_0$ and the noise derivative $\xi(\nu)$ at time $\nu$, also integrated from $t_0$ to $t$ of the variable $\nu$. We will later see that (3) appears again in the Kolmogorov equation.

**Theorem 1.**

$$\Gamma(t, t_0) = \widetilde{\Gamma}(t, t_0) \tag{4}$$

*Proof* 1.

$$
\begin{aligned}
\Gamma(t,t_0) &= \int_{t_0}^{t} E(\xi(t)\xi(\nu))d\nu = \frac{\sigma^2}{2\tau}\int_{t_0}^{t}\Pi(|t-\nu|) \\
&\cdot d\nu = \frac{\sigma^2}{2\tau}\int_{t-t_0}^{0}\Pi(|z|)dz \\
\widetilde{\Gamma}(t,t_0) &= \int_{t_0}^{t} E(\xi(t_0)\xi(\nu))d\nu = \frac{\sigma^2}{2\tau}\int_{t_0}^{t}\Pi(|t_0-\nu|) \\
&\cdot d\nu = \frac{\sigma^2}{2\tau}\int_{0}^{t-t_0}\Pi(|z|)dz = \Gamma(t,t_0), \text{Qed.}
\end{aligned}
\tag{5}
$$

The equality of $\Gamma(t,t_0)$ and $\widetilde{\Gamma}(t,t_0)$ in Theorem 1 means that the integral from $t_0$ to $t$ of the expected value of the product of the two noise derivatives does not depend on whether the first noise derivative is specified at time $t$ as $\xi(t)$ or is specified at the initial time $t_0$ as $\xi(t_0)$.

An example to be studied in this article is

$$
\begin{aligned}
\Pi\left(\left|t-t'\right|\right) &\equiv \Pi_1\left(\left|t-t'\right|\right)\Pi_2\left(\left|t-t'\right|\right), \\
\Pi_1\left(\left|t-t'\right|\right) &\equiv \mathrm{Exp}\left(-\frac{\left|t-t'\right|}{\tau}\right), \\
\Pi_2\left(\left|t-t'\right|\right) &\equiv \left(\frac{\left|t-t'\right|}{\beta}\right)^{2c-1}
\end{aligned}
\tag{6}
$$

where $c>0$ is a parameter. The exponential form $\Pi_1(|t-t'|)$, which is assumed to be well-behaved for all $t$ and $t'$, is known as noise if it is Gaussian distributed. When $\tau$ approaches zero in (6), the well-known Dirac delta function $\delta$ () applies for $\Pi_1(|u|)/(2\tau)$ since $\mathrm{Lim}_{\tau\longrightarrow 0}\Pi_1(|u|)/(2\tau)=\delta(u)$. $\Pi_2(|t-t'|)$ relates to fractional Brownian motion. Unless $c\geq 1/2$, $\Pi_2(|t-t'|)$ is singular at $t=t'$.

Inserting (2) and (6) into (3) gives

$$
\begin{aligned}
\Gamma(t,t_0) &= \frac{\sigma^2}{2\tau}\int_{t_0}^{t}\Pi_1(|t-\nu|)\Pi_2(|t-\nu|)d\nu \\
&= \frac{\sigma^2}{2\tau}\int_{t_0-t}^{0}\mathrm{Exp}\left(\frac{z}{\tau}\right)\left(\frac{-z}{\beta}\right)^{2c-1}dz \\
&= \frac{\sigma^2}{2}\frac{\tau^{2c-1}}{\beta^{2c-1}}\left(\Gamma_f(2c,0)-\Gamma_f\left(2c,\frac{(t-t_0)}{\tau}\right)\right), c>0
\end{aligned}
\tag{7}
$$

where the well-known Gamma function is $\Gamma_f(z,z')\equiv\int_{z'}^{\infty}\mathrm{Exp}(-u)u^{z-1}du$.

Determining the limit when $\tau\longrightarrow 0$ in (7) implies

$$
\mathrm{Lim}_{\tau\longrightarrow 0}\Gamma(t,t_0) = \frac{\sigma^2}{2}\frac{\tau^{2c-1}}{\beta^{2c-1}}\Gamma_f(2c,0) = \frac{\sigma^2}{2}\frac{\tau^{2c-1}}{\beta^{2c-1}}\int_0^{\infty}\mathrm{Exp}(-u)u^{2c-1}du
\tag{8}
$$

which equals $\sigma^2/2$ when $c=1/2$, since $\Gamma_f(1,0)=1$ and $\tau^{2c-1}/\beta^{2c-1}=1$.

Inserting $c=1/2$ into (7) gives

$$
\begin{aligned}
\Gamma(t,t_0) &= \frac{\sigma^2}{2}\left(\Gamma_f(1,0)-\Gamma_f\left(1,\frac{(t-t_0)}{\tau}\right)\right) \\
&= \frac{\sigma^2}{2}\left(1-\mathrm{Exp}\left(\frac{-(t-t_0)}{\tau}\right)\right)
\end{aligned}
\tag{9}
$$

since $\Gamma_f(1,(t-t_0/\tau))=\mathrm{Exp}(-(t-t_0)/\tau)$

Inserting $c=1$ into (7) gives

$$
\Gamma(t,t_0) = \frac{\sigma^2}{2}\frac{\tau}{\beta}\left(1-\Gamma_f\left(2,\frac{t-t_0}{\tau}\right)\right)
\tag{10}
$$

since $\Gamma_f(2,0)=1$.

Let us define

$$
\Omega^{(2)}(t,t_0) \equiv \int_{t_0}^{t}\int_{t_0}^{t}E(\xi(u)\xi(\nu))\mathrm{dudv}
\tag{11}
$$

which expresses the variance of the noise derivative $\xi(t)$ at time $t$. Inserting the equations above into (11) gives

$$
\begin{aligned}
\Omega^{(2)}(t,t_0) &= \frac{\sigma^2}{2\tau}\frac{1}{\beta^{2c-1}}\int_{t_0}^{t}\int_{t_0}^{t}\mathrm{Exp}\left(-\frac{|u-\nu|}{\tau}\right)|u-\nu|^{2c-1}dud\nu \\
&= \frac{\sigma^2}{2\tau}\frac{1}{\beta^{2c-1}}\int_{t_0}^{t}\int_{t_0-\nu}^{t-\nu}\mathrm{Exp}\left(-\frac{|z|}{\tau}\right)|z|^{2c-1}dzd\nu \\
&= \frac{\sigma^2}{2\tau}\frac{1}{\beta^{2c-1}}\int_{t_0}^{t}\left[\int_{t_0-\nu}^{0}(-z)^{2c-1}\mathrm{Exp}\left(\frac{z}{\tau}\right)dz+\int_0^{t-\nu}z^{2c-1}\mathrm{Exp}\left(-\frac{z}{\tau}\right)dz\right]d\nu \\
&= \frac{\sigma^2}{2}\frac{\tau^{2c-1}}{\beta^{2c-1}}\int_{t_0}^{t}\left[\Gamma_f(2c,0)-\Gamma_f\left(2c,\frac{\nu-t_0}{\tau}\right)+\Gamma_f(2c,0)-\Gamma_f\left(2c,\frac{t-\nu}{\tau}\right)\right]d\nu \\
&= \sigma^2\frac{\tau^{2c-1}}{\beta^{2c-1}}\int_{t_0}^{t}\left[\Gamma_f(2c,0)-\Gamma_f\left(2c,\frac{(\nu-t_0)}{\tau}\right)\right]d\nu
\end{aligned}
\tag{12}
$$

Analytical integration of the $\Gamma_f(\quad)$ is not possible unless $2c$ is an integer. See Appendix C for examples when $c=1/2$ and $c=1$.

Denote $\partial\Omega^2(t,t_0)/\partial t_0$ by $D_2\Omega^2(t,t_0)$. □

**Theorem 2.**

$$
D_2\Omega^{(2)}(t,t_0) = -2\widetilde{\Gamma}(t,t_0)
\tag{13}
$$

*Proof* 2.

$$
\begin{aligned}
\Omega^{(2)}(t,t_0) &= \int_{t_0}^{t}\int_{t_0}^{t}E(\xi(u)\xi(\nu))dud\nu = E\left(\left[\int_{t_0}^{t}\xi(\nu)d\nu\right]^2\right) \\
D_2\Omega^{(2)}(t,t_0) &= -2E\left(\int_{t_0}^{t}\xi(t_0)\xi(\nu)d\nu\right) = -2\widetilde{\Gamma}(t,t_0), \text{Qed.}
\end{aligned}
\tag{14}
$$

For a comparison with fractional Brownian motion [19], it is useful to define what can be seen as the correlation of $\zeta(t)$, to read

$$\Omega^{(2)}\left(t, t', t_0\right) \equiv \int_{t_0}^{t'}\int_{t_0}^{t} E(\xi(u)\xi(v))dudv \tag{15}$$

where $\partial^2\Omega^{(2)}(t, t', t_0)/\partial t\partial t' = E(\xi(t)\xi(t'))$.

Assuming $t \geq t'$, general $c > 0$, and inserting fractional Brownian motion with $\Pi_1(|t-t'|) = 1$ into (12) gives

$$\begin{aligned}
\Omega^{(2)}\left(t, t', t_0\right) &= \frac{\sigma^2}{2\tau}\frac{1}{\beta^{2c-1}}\int_{t_0}^{t'}\int_{t_0}^{t}|u-v|^{2c-1}dudv \\
&= \frac{\sigma^2}{2\tau}\frac{1}{\beta^{2c-1}}\int_{t_0}^{t'}\int_{t_0-v}^{t-v}|z|^{2c-1}dzdv \\
&= \frac{\sigma^2}{2\tau}\frac{1}{\beta^{2c-1}}\int_{t_0}^{t'}\left[\int_{t_0-v}^{0}(-z)^{2c-1}dz + \int_{0}^{t-v}z^{2c-1}dz\right]dv \\
&= \frac{\sigma^2}{2\tau}\frac{1}{\beta^{2c-1}}\frac{1}{2c}\int_{t_0}^{t'}\left[\left((v-t_0)^{2c} + (t-v)^{2c}\right)\right]dv \\
&= \frac{\sigma^2}{2\tau}\frac{1}{\beta^{2c-1}}\frac{1}{2c(2c+1)} \\
&\quad\cdot\left((t-t_0)^{2c+1} + \left(t'-t_0\right)^{2c+1} - \left(t-t'\right)^{2c+1}\right), t \geq t' \geq t_0
\end{aligned} \tag{16}$$

Assuming $c > 0$ is necessary to achieve a converging integral in (16). In general, when $t \geq t_0$ and $t' \geq t_0$, (16) can be rewritten as

$$\begin{aligned}
\Omega^{(2)}\left(t, t', t_0\right) &= \frac{\sigma^2}{2\tau}\frac{1}{\beta^{2H-2}}\frac{1}{(2H-1)2H} \\
&\quad\cdot\left((t-t_0)^{2H} + \left(t'-t_0\right)^{2H} - \left|t-t'\right|^{2H}\right), H > \frac{1}{2}
\end{aligned} \tag{17}$$

where $H \equiv c + 1/2$ is the Hurst parameter where $c > 0 \Rightarrow H > 1/2$. An alternative to (17) is

$$\Omega^{(2)}\left(t, t', t_0\right) \equiv \lambda\left((t-t_0)^{2H} + \left(t'-t_0\right)^{2H} - \left|t-t'\right|^{2H}\right) \tag{18}$$

where $\lambda > 0$ is a parameter. Differentiating (18) with respect to $t$ and $t'$ gives

$$\frac{\partial^2\Omega^{(2)}\left(t, t', t_0\right)}{\partial t\partial t'} = \lambda\, 2H(2H-1)\left|t-t'\right|^{2H-2} \tag{19}$$

which is singular when $t = t'$ unless $H \geq 1/2$. Thus, when $H \leq 1/2$, the variance of the noise derivative $\xi(t)$ when $t = t'$ is infinite due to division with 0 in (19). Summing up, the above development shows that Brownian motion is easily modelled by the exponential correlation function $\mathrm{Exp}(-(t-t_0)/\tau)$ in (13) at the limit when $\tau \longrightarrow 0$.

Equation (17) corresponds to fractional Brownian motion if the noise derivative $\xi(t)$ is Gaussian. Let

$$\begin{aligned}
&\Omega^{(1)}(t, t_0) \equiv \int_{t_0}^{t} E(\xi(u))du = 0 \\
&\Omega^{(2)}(t, t_0) \equiv \int_{t_0}^{t}\int_{t_0}^{t} \mathrm{E}(\xi(\mathrm{u})\xi(\mathrm{v}))dudv \\
&\Omega^{(2n-1)}(t, t_0) \equiv \int_{t_0}^{t}\int_{t_0}^{t}\int_{t_0}^{t}\cdots\int_{t_0}^{t} E(\xi(u_1)\xi(u_2)\cdots\xi(u_{2n-1}))du_1du_2du_3\cdots du_{2n-1}, n = 1, 2, 3\cdots \\
&\Omega^{(2n)}(t, t_0) \equiv \int_{t_0}^{t}\int_{t_0}^{t}\int_{t_0}^{t}\cdots\int_{t_0}^{t} E(\xi(u_1)\xi(u_2)\cdots\xi(u_{2n}))du_1du_2du_3\cdots du_{2n}, n = 1, 2, 3\cdots
\end{aligned} \tag{20}$$

Assuming Gaussian noise $\zeta(t)$ in (20) implies

$$\begin{aligned}
&\Omega^{(2n-1)}(t, t_0) = 0, \Omega^{(2n)}(t, t_0) = \frac{(2n)!}{n!}\left(\frac{\Omega^{(2)}(t, t_0)}{2}\right)^n, n = 1, 2, 3\cdots \\
&\Omega^{(2)}(t, t_0) = \sigma^2\frac{\tau}{\beta}\left[t - t_0 - 2\tau + \mathrm{Exp}\left(-\frac{t-t_0}{\tau}\right)((t-t_0) + 2\tau)\right], c = 1 \\
&\sigma^2\frac{\tau}{\beta}\left[t - t_0 - 2\tau + ((t-t_0) + 2\tau)\left(1 - \frac{(t-t_0)}{\tau} + \frac{(t-t_0)^2}{(2\tau^2)} - \frac{(t-t_0)^3}{(6\tau^3)}\right)\right] \\
&= \sigma^2\frac{\tau}{\beta}\left[t - t_0 - 2\tau + (t-\mathrm{t}_0)\left(1 - \frac{(t-t_0)}{\tau} + \frac{(t-t_0)^2}{(2\tau^2)}\right) + 2\tau\left(1 - \frac{(t-t_0)}{\tau} + \frac{(t-t_0)^2}{(2\tau^2)} - \frac{(t-t_0)^3}{(6\tau^3)}\right)\right] \\
&= \sigma^2\frac{\tau}{\beta}\left[t - t_0 - 2\tau + (t-t_0) - \frac{(t-t_0)^2}{\tau} + 2\tau - 2(t-t_0) + \frac{(t-t_0)^2}{(\tau)} + \frac{(t-t_0)^3}{(2\tau^2)} - \frac{(t-t_0)^3}{(3\tau^2)}\right] \\
&= \sigma^2\frac{\tau}{\beta}\left[\frac{(t-t_0)^3}{(6\tau^2)}\right]
\end{aligned} \tag{21}$$

□

## 3. The Partial Differential Equation for Probability Based on Differentiable Noise

The probability density is denoted as $\rho(t, t_0, s_0, s)$ where $\rho(t, t_0, s_0, s)ds$ is the probability that the asset value $S(t)$ is within the interval $(s, s + \mathrm{d}s)$ at time $t$, $t_0$ is the initial time, and $S(t_0, t_0, s_0) = s_0$ is the initial asset value at time $t_0$. Assume without loss of generality that

$$\rho(t, t_0, s_0, s) = \int_{-\infty}^{\infty}\rho(t, t_0, s_0, u)\delta(s-u)du \tag{22}$$

which expresses that the probability density $\rho(t, t_0, s_0, s)$ equals the probability density $\rho(t, t_0, s_0, u)$ multiplied with the Dirac delta function $\delta(s-u)$ integrating the real variable $u$ from minus infinity to plus infinity.

Assuming differentiable noise $\zeta(t)$ in (22) implies

$$\begin{aligned}
&\frac{d\rho(t, t_0, s_0, S(t))}{dt} = D_1(t, S(t)) + D_2\rho(t, t_0, s_0, S(t))\dot{S}(t) = 0 \\
&\Leftrightarrow D_1\rho(t, t_0, s_0, S(t)) = -D_2\rho(t, t_0, s_0, S(t))\dot{S}(t)
\end{aligned} \tag{23}$$

where $D_i, i = 1, 2$, denotes the derivative of the ith argument. Conservation of the probability $\rho(t, t_0, s_0, s)$ in (22) implies

$$\frac{\partial\rho(t, t_0, s_0, s)}{\partial t} = -\frac{\partial}{\partial s}(\rho(t, t_0, s_0, s)v(t, s)) \tag{24}$$

where $v(t, s)$ is a real, continuous, and differentiable function with respect to time $t$ and asset value $s$. Assuming $v(t, s) = \dot{S}(t)$ as in (1) and $s = S(t)$ imply

$$v(t, s) = G_1(t, s) + G_2(t, s)\xi(t) \tag{25}$$

Inserting (25) into (24) gives

$$\begin{aligned}\frac{\partial \rho(t, t_0, s_0, s)}{\partial t} &= -\frac{\partial}{\partial s}(\rho(t, t_0, s_0, s)G_1(t, s)) \\ &\quad - \frac{\partial}{\partial s}(\rho(t, t_0, s_0, s)G_2(t, s)\xi(t)) \\ &= -\frac{\partial}{\partial s}(\rho(t, t_0, s_0, s)G_1(t, s)) \\ &\quad - \frac{\partial}{\partial s}\int_{-\infty}^{\infty} \rho(t, t_0, s_0, u)G_2(t, u)\xi(t)\delta(s-u)du\end{aligned} \tag{26}$$

with functional derivative $\delta S(t)/\delta\xi(t) = G_2(t, s)$.

To develop the last term on the right-hand side in (26), Novikov's [7] identity is used. It assumes the Gaussian noise derivative $\xi(t)$ and that $G_2(t, s) = G_2(t)$ depends on time $t$ only. Then, $G_2(t)$ can be placed outside the integral in (26) which implies

$$E(G_2(t)\xi(t)\delta(s - S(t))) = \int_{t_0}^{t} E(\xi(t)\xi(v))dv \times E\left(\frac{\delta}{\delta\xi}[G_2(t)\delta(s - S(t))]\right) \tag{27}$$

where

$$\begin{aligned}\frac{\delta}{\delta\xi}[G_2(t)\delta(s - S(t))] &= G_2(t)\frac{\delta}{\delta\xi}\delta(s - S(t)) \\ &= G_2(t)\frac{\partial}{\partial S(t)}\delta(s - S(t))\frac{\delta S}{\delta\xi} \\ &= -G_2(t)^2\frac{\partial}{\partial s}\delta(s - S(t))\end{aligned} \tag{28}$$

In Theorem 3, we generalize the identity to apply more generally when $G_2(t, S(t))$ is a function of both time $t$ and asset value $S(t)$.

**Theorem 3.** *For Gaussian noise $\zeta(t)$,*

$$\begin{aligned}&E(G_2(t, S(t))\xi(t)\delta(s - S(t))) \\ &\quad = \int_{t_0}^{t} E(\xi(t)\xi(v))dv \times E\left(\frac{\delta}{\delta\xi}[G_2(t, S(t))\delta(s - S(t))]\right)\end{aligned} \tag{29}$$

*Proof* 3. See Appendix D.

Differentiating the expression in Theorem 3 with respect to the noise derivative $\xi(t)$ gives

$$\begin{aligned}&\frac{\delta}{\delta\xi}(G_2(t, S(t))\delta(s - S(t))) \\ &= \frac{\partial G_2(t, S(t))}{\partial S(t)}\frac{\delta S}{\delta\xi}\delta(s - S(t)) + G_2(t, S(t))\frac{\partial\delta(s - S(t))}{\partial S(t)}\frac{\delta S}{\delta\xi} \\ &= \frac{\partial G_2(t, S(t))}{\partial S(t)}G_2(t, S(t))\delta(s - S(t)) - G_2(t, S(t))^2\frac{\partial}{\partial s}\delta(s - S(t))\end{aligned} \tag{30}$$

which implies

$$\begin{aligned}&-\frac{\partial}{\partial s}\int_{-\infty}^{\infty}\rho(t, t_0, s_0, u)G_2(t, u)\xi(t)\delta(s-u)du \\ &= -\Gamma(t,t_0)\frac{\partial}{\partial s}\int_{-\infty}^{\infty}\rho(t, t_0, s_0, u)\frac{\partial}{\partial u}\left(\frac{G_2(t, u)^2}{2}\right)\delta(s-u)du \\ &\quad + \Gamma(t,t_0)\frac{\partial^2}{\partial s^2}\int_{-\infty}^{\infty}\rho(t, t_0, s_0, u)G_2(t, u)^2\delta(s-u)du \\ &= -\Gamma(t, t_0)\frac{\partial}{\partial s}\left(\rho(t, t_0, s_0, s)\frac{\partial}{\partial s}\left(\frac{G_2(t, s)^2}{2}\right)\right) \\ &\quad + \Gamma(t, t_0)\frac{\partial^2}{\partial s^2}\left(\rho(t, t_0, s_0, s)G_2(t, s)^2\right)\end{aligned} \tag{31}$$

Assuming correlated differentiable Gaussian noise $\zeta(t)$ and inserting (31), the generalized Fokker–Planck equation (or Kolmogorov forward equation) [8–10] becomes

$$\begin{aligned}\frac{\partial\rho(t, t_0, s_0, s)}{\partial t} &= -\frac{\partial}{\partial s}\left(\left(G_1(t, s) + \frac{\Gamma(t, t_0)}{2}\frac{\partial}{\partial s}\left(G_2(t, s)^2\right)\right)\rho(t, t_0, s_0, s)\right) \\ &\quad + \Gamma(t, t_0)\frac{\partial^2}{\partial s^2}\left(G_2(t, s)^2\rho(t, t_0, s_0, s)\right)\end{aligned} \tag{32}$$

which applies $\Gamma(t, t_0)$ in (3). The generalized Kolmogorov [8] backward equation becomes (Appendix E)

$$\begin{aligned}-\frac{\partial\rho(t, t_0, s_0, s)}{\partial t_0} &= \left(G_1(t_0, s_0) + \frac{\widetilde{\Gamma}(t, t_0)}{2}\frac{\partial}{\partial s_0}\left(G_2(t_0, s_0)^2\right)\right) \\ &\quad \cdot \frac{\partial\rho(t, t_0, s_0, s)}{\partial s_0} + \widetilde{\Gamma}(t, t_0)G_2(t, s_0)^2\frac{\partial^2\rho(t, t_0, s_0, s)}{\partial {s_0}^2}\end{aligned} \tag{33}$$

which applies $\widetilde{\Gamma}(t, t_0)$ in (3). The differentiable noise $\zeta(t)$ can be set approximately equal to any nondifferentiable noise $\Sigma_t$, in the sense that $\Sigma_t^{(n)}$ is a $t$-continuous differential process such that for almost all initial conditions, $\lim_{n\longrightarrow\infty}\Sigma_t^{(n)} = \Sigma_t$, uniformly in bounded intervals, $\xi(t) \equiv \partial\Sigma_t^{(n)}/\partial t$. Of interest is the Gaussian–Wiener noise/process. When the nondifferentiable noise $\Sigma_t = W_t$, where $W_t$ is the Wiener noise at time $t$, the Ito [2] integral, or the Fisk–Stratonovich [13, 14] integrals have been used in the literature. Sussmann [15] shows that for the Wiener noise, the solution, based on the Fisk-Stratonovich [13, 14] integrals, coincides with the solution obtained when $\xi(t) = \lim_{n\longrightarrow\infty} \partial\Sigma_t^n/\partial t$. Thus, (33) is also valid when applying the Fisk–Stratonovich [13, 14] integrals within the Wiener noise/process [15]. See Appendix F for Riemann–Stieltjes, Ito [2], and Fisk–Stratonovich [13, 14] integrals. □

## 4. Generalizing Black and Scholes' [1] Partial Differential Equation

Whereas Black and Scholes [1] assume nondifferentiable noise, this section develops an alternative equation assuming differentiable noise $\xi(t)$. Consider the expected value $E(f(t, t_0, S(t, t_0, s_0)))$ of some function $f(t, t_0, S(t, t_0, s_0))$ to be specified, that is,

$$E_V(t, t_0, s_0) \equiv E(f(t, t_0, S(t, t_0, s_0))) = \int_{-\infty}^{\infty} \rho(t, t_0, s_0, s) f(t, t_0, s) ds,$$

$$V(t, t_0, s_0) \equiv f(t, t_0, S(t, t_0, s_0)), \tag{34}$$

where $\rho(t, t_0, s_0, s)$ is defined in the previous section and $s = S(t, t_0, s_0)$. Assuming correlated Gaussian noise $\zeta(t)$, assuming $\widetilde{\Gamma}(t, t_0) = \int_{t_0}^{t} E(\xi(t_0)\xi(v))dv$ defined in (3), and differentiating (34) with respect to the initial time $t_0$ give

$$\frac{\partial E_V(t, t_0, s_0)}{\partial t_0} = \int_{-\infty}^{\infty} \rho(t, t_0, s_0, s) \frac{\partial f(t, t_0, s)}{\partial t_0} \cdot ds + \int_{-\infty}^{\infty} \frac{\partial \rho(t, t_0, s_0, s)}{\partial t_0} f(t, t_0, s) ds \tag{35}$$

which is rewritten as

$$\begin{aligned} &-\frac{\partial E_V(t, t_0, s_0)}{\partial t_0} + \int_{-\infty}^{\infty} \rho(t, t_0, s_0, s) \frac{\partial f(t, t_0, s)}{\partial t_0} ds \\ &= -\int_{-\infty}^{\infty} \frac{\partial \rho(t, t_0, s_0, s)}{\partial t_0} f(t, t_0, s) ds \\ &= \int_{-\infty}^{\infty} \Bigg[ \left( G_1(t_0, s_0) + \frac{\widetilde{\Gamma}(t, t_0)}{2} \frac{\partial}{\partial s_0} \left(G_2(t_0, s_0)^2\right) \right) \frac{\partial \rho(t, t_0, s_0, s)}{\partial s_0} \\ &\quad + \widetilde{\Gamma}(t, t_0) G_2(t_0, s_0)^2 \frac{\partial^2 \rho(t, t_0, s_0, s)}{\partial {s_0}^2} \Bigg] f(t, t_0, s) ds \\ &= \left[ \left( G_1(t_0, s_0) + \frac{\widetilde{\Gamma}(t, t_0)}{2} \frac{\partial}{\partial s_0} \left(G_2(t_0, s_0)^2\right) \right) \frac{\partial}{\partial s_0} + \widetilde{\Gamma}(t, t_0) G_2(t_0, s_0)^2 \frac{\partial^2}{\partial {s_0}^2} \right] E_V(t, t_0, s_0) \end{aligned} \tag{36}$$

which is rewritten as

$$\frac{\partial E_V(t, t_0, s_0)}{\partial t_0} + \left( G_1(t_0, s_0) + \frac{\widetilde{\Gamma}(t, t_0)}{2} \frac{\partial}{\partial s_0} \left(G_2(t_0, s_0)^2\right) \right) \frac{\partial E_V}{\partial s_0} + \widetilde{\Gamma}(t, t_0) G_2(t_0, s_0)^2 \frac{\partial^2 E_V}{\partial {s_0}^2} = \int_{-\infty}^{\infty} \rho(t, t_0, s_0, s) \frac{\partial f}{\partial t_0} ds \tag{37}$$

Inserting the asset value $V = S(t, t_0, s_0)$ and the functions $G_1 = rs$ and $G_2 = s$, where $r > 0$ is a parameter, into (37), gives

$$\frac{\partial E_V}{\partial t_0} + \left( rs_0 + \widetilde{\Gamma}(t, t_0) s_0 \right) \frac{\partial E_V}{\partial s_0} + \widetilde{\Gamma}(t, t_0) s_0^2 \frac{\partial^2 E_V}{\partial {s_0}^2} = 0 \tag{38}$$

As an alternative, inserting $V = S(t, t_0, s_0)\mathrm{Exp}(-r(t - t_0))$, which expresses depreciating asset value, the functions $G_1 = rs$ and $G_2 = s$, and $c = 1/2$ into (37), assuming constant correlation strength $\sigma^2$, and considering the limit when the correlation time $\tau \longrightarrow 0$ approaches zero, give

$$\frac{\partial E_V}{\partial t_0} + rs_0 \frac{\partial E_V}{\partial s_0} + \frac{\sigma^2}{2} s_0 \frac{\partial E_V}{\partial s_0} + \frac{\sigma^2}{2} s_0^2 \frac{\partial^2 E_V}{\partial {s_0}^2} = rE_V \tag{39}$$

The term $(\sigma^2/2)s_0(\partial E_V/\partial s_0)$ in (39) is absent in the original Black and Scholes [1] equation which is based on nondifferentiable white noise and the Ito [2] stochastic integral. However, applying the Fisk–Stratonovich [13, 14] integrals would have led to (39) [15]. Solving (39) gives

$$\begin{aligned} E_V(t_0, s_0) = \frac{1}{2} e^{-rt - t_0\sigma^2/2} \Bigg( &-e^{t_0\left(r+\sigma^2/2\right)} K\, Erfc\left[\frac{-rt + rt_0 + Ln[K] - Ln[s_0]}{\sigma\sqrt{2}\sqrt{t - t_0}}\right] \\ &+ e^{t\left(r+\sigma^2/2\right)} s_0 Erfc\left[\frac{-\left(\left(t - t_0\right)\left(r + \sigma^2\right)\right) + \mathrm{Ln}[K] - \mathrm{Ln}[s_0]}{\sigma\sqrt{2}\sqrt{t - t_0}}\right] \Bigg), E_V(t, s_0) \\ = \mathrm{Max}[s_0 - K, 0] \end{aligned} \tag{40}$$

where $Ln[z]$ is the natural logarithm of $z$ and $Erfc[z]$ is the complementary error function of $z$.

**Example 1.** Consider the value $E_V$ of a European vanilla call option where the underlying asset price $s_0$ and the strike price $K$ are both $s_0 = K = \$100$, the annualized risk-free interest rate is $r = 5\%$, the volatility of the underlying asset (i.e., the standard deviation of the stock's returns, i.e. the square root of the constant correlation strength) is $\sigma = 20\%$, the maturity period (i.e., time of option expiration) is $t = 1$ year, and the time until maturity is $t - t_0 = 1$ year where $t_0 = 0$. Inserting into (40) gives $E_V = 11.7746$. However, inserting the same values into the Black and Scholes [1], where the term $(\sigma^2/2)s_0(\partial E_V/\partial s_0)$ in (39) is absent causing a slightly different (40), gives the lower value $E_{VBS} = 10.4506$, where the subscripts BS denote Black and Scholes [1]. Using the Mathematica 13 software package (http://www.wolfram.com), Figure 1 plots the values $E_{\mathrm{VBS}}$ and $E_V$ as functions of each of the six parameters $s_0$, $K$, $\sigma$, $t$, $r$, and $t_0$, keeping the other five parameter values at their benchmarks specified above and marked with a vertical dashed line in each panel.

The value $E_V$ according to (40) exceeds the Black and Scholes [1] value $E_{\mathrm{VBS}}$ throughout. In Figure 1(a), $E_V$ and $E_{VBS}$ increase in $s_0$, while in Figure 1(b), they decrease in $K$. In Figures 1(c), 1(d), and 1(e), $E_V$ and $E_{VBS}$ increase in $\sigma$, $t$, and $r$. Substantial increases in the standard deviation $\sigma$ and the interest rate $r$ cause increasing discrepancies between $E_V$ and $E_{VBS}$. In Figure 1(f), $E_V$ and $E_{VBS}$ decrease in $t_0$ reaching $E_V = E_{VBS} = 0$ when $t_0 = t = 1$.

The value of futures in finance is $V = (S(t, t_0, s_0) - K)\mathrm{Exp}(-r(t - t_0))$, where $K \geq 0$ is the constant cost of

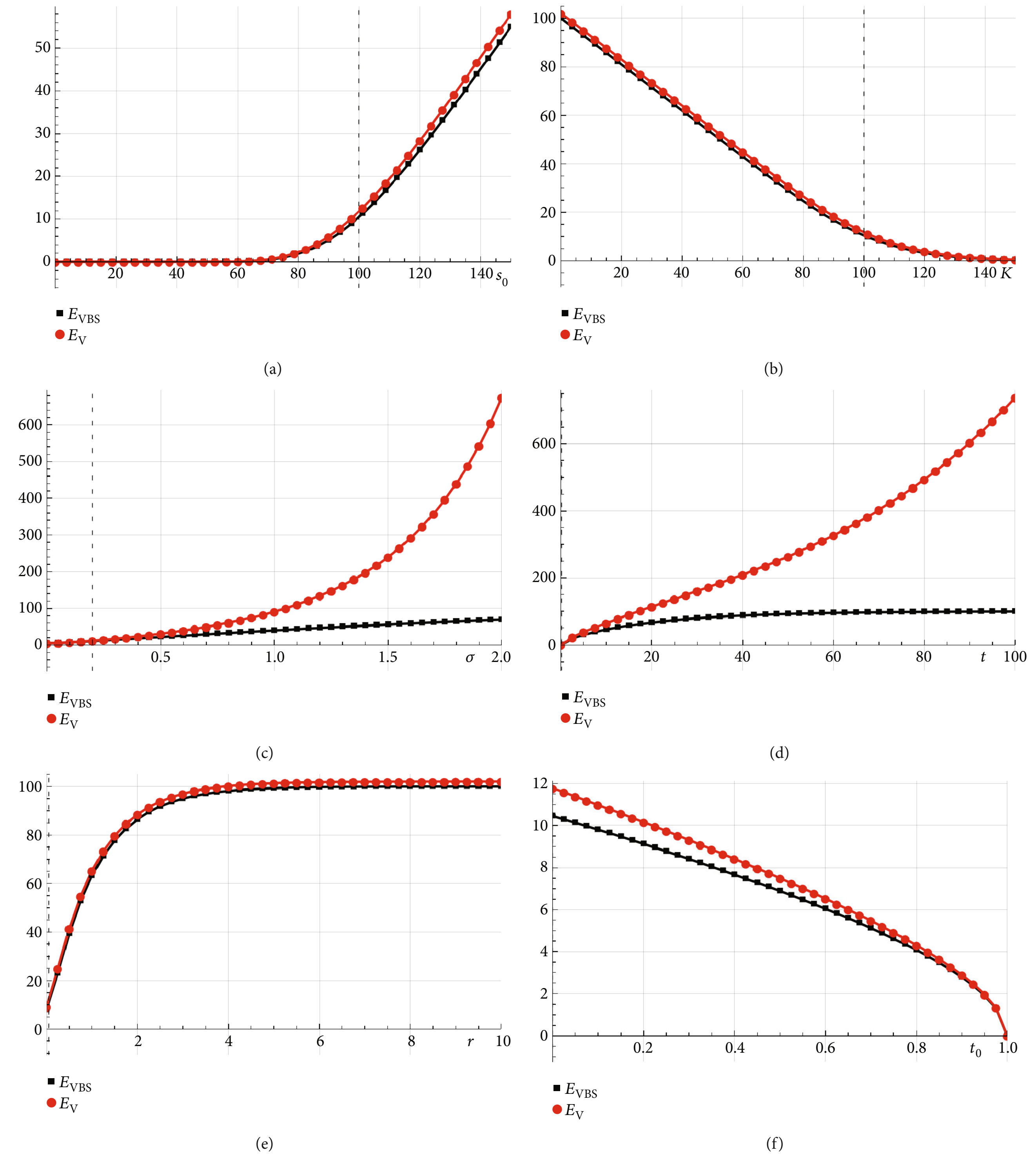


FIGURE 1: The values $E_{VBS}$ and $E_V$ as functions of each of the six parameters (a) $s_0$, (b) $K$, (c–e) $\sigma$, $t$, and $r$, and (f) $t_0$, relative to the benchmark parameter values $s_0 = K = \$100$, $\sigma = 20\%$, $t = 1$, $r = 5\%$, and $t_0 = 0$.

purchasing the futures which is subtracted from the asset value $S(t, t_0, s_0)$, which satisfies (39) since $\partial(S(t, t_0, s_0) - K)\text{Exp}(-r(t - t_0))/\partial t_0 = r(S(t, t_0, s_0) - K)\text{Exp}(-r(t - t_0))$. However, $V = (S(t, t_0, s_0) - K(t_0))\text{Exp}(-r(t - t_0))$ does not satisfy (39) since a cost $K(t_0)$ of purchasing the futures dependent on the initial asset value $S(t, t_0, s_0)$ at time $t_0$ violates (39).

Assume the asset value

$$V = f(t, t_0, S(t, t_0, s_0)) = \text{Exp}\left(-\int_{t_0}^{t} q(t, u)du\right)H(S(t, t_0, s_0)) \tag{41}$$

where $H(S(t,t_0,s_0))$ and $q(t,u)$ are arbitrary functions. Equation (41) can be interpreted as an extended Feynman–Kac [28] equation. Applying $\partial f/\partial t_0 = f\, q(t,t_0)$, inserting into (37) gives

$$\frac{\partial E_V}{\partial t_0} + \left( G_1(t_0,s_0) + \frac{\widetilde{\Gamma}(t,t_0)}{2}\frac{\partial}{\partial s_0}\left(G_2(t_0,s_0)^2\right)\right)\frac{\partial E_V}{\partial s_0} + \widetilde{\Gamma}(t,t_0)G_2(t_0,s_0)^2\frac{\partial^2 E_V}{\partial {s_0}^2} = q(t,t_0)E_V \tag{42}$$

## 5. The Random Draw Gives an Equivalent and Associated Stochastic Process

As in (1) and (2), let $S(t)$ be given by the differential equation with differentiable noise $\zeta(t)$, that is,

$$\dot{S}(t) = G_1(t,S(t)) + G_2(t,S(t))\xi(t), E(\xi(t)) = 0, E\left(\xi(t)\xi\left(t'\right)\right) = \frac{\sigma^2}{2\tau}\Pi\left(\left|t-t'\right|\right) \tag{43}$$

This section mimics $\xi(t)$ and (43) applying a nondifferentiable noise interpreted as a random draw, which can be useful in numerical applications.

Assume that (43) is realized through the Euler [32] scheme

$$S_{t+h} = S_t + \left(G_1(t,S_t) + \frac{\Gamma(t,t_0)}{2}\frac{\partial\left(G_2(t,S_t)^2\right)}{\partial S_t}\right)h + G_2(t,S_t)d*w_t,$$
$$d*w_t = R_d(0,2\Gamma(t,t_0)h), \Gamma(t,t_0) = \int_{t_0}^{t} E(\xi(t)\xi(v))dv \tag{44}$$

where $d*w_t$ expresses a random draw from some distribution $d$ with expectation zero and variance $2\widetilde{\Gamma}h$ and $h$ is the time step. Consider an arbitrary function $H(\,)$. The random draw means that the noise is nondifferentiable and without bounded variation, that is, $E(H(S_t)R_d(0,2\Gamma h)) = 0$, while $E(H(S_t)R_d(0,2\Gamma h)^2) = E(H(S_t)2\Gamma h)$. Taylor expansion implies

$$\begin{aligned} H(S_{t+h}) &= H\left(S_t + \left(G_1(t,S_t) + \frac{\Gamma(t,t_0)}{2}\frac{\partial\left(G_2(t,S_t)^2\right)}{\partial S_t}\right)h + G_2(t,S_t)d*w_t\right) \\ &= H(S_t) + H'(S_t)\left(\left(G_1(t,S_t) + \frac{\Gamma(t,t_0)}{2}\frac{\partial\left(G_2(t,S_t)^2\right)}{\partial S_t}\right)h + G_2(t,S_t)d*w_t\right) \\ &\quad + \frac{1}{2}H''(S_t)\left(2\left(G_1(t,S_t) + \frac{\Gamma(t,t_0)}{2}\frac{\partial\left(G_2(t,S_t)^2\right)}{\partial S_t}\right)G_2(t,S_t)d*w_t h + G_2(t,S_t)^2 d*{w_t}^2\right) \\ &\quad + O(h^2) \end{aligned} \tag{45}$$

where $H'(S_t)$ and $H''(S_t)$ mean derivative and double derivative, respectively, with respect to the asset value $S_t$ at time $t$. Moments of higher order than two for the distribution $d$ are assumed to be of order $h$ larger than one, which applies for the Gaussian distribution. Taking the expectation $E(H(S_t))$ of (45) and differentiating with respect to time $t$, which involves considering the limit as the time step $h$ approaches zero, give

$$\frac{d}{dt}E(H(S_t)) = E\left(\left(G_1(t,S_t) + \frac{\Gamma(t,t_0)}{2}\frac{\partial\left(G_2(t,S_t)^2\right)}{\partial S_t}\right)H'(S_t)\right) + E\left(\Gamma(t,t_0)G_2(t,S_t)^2H''(S_t)\right) \tag{46}$$

Applying partial integration of (46) and assuming that the density $\rho(t,t_0,s_0,s)$ is zero at the boundaries $s=-\infty$ and $s=\infty$ imply

$$\begin{aligned} \frac{d}{dt}\int_{-\infty}^{\infty}\rho(t,t_0,s_0,s)H(s)ds &= \int_{-\infty}^{\infty}\dot{\rho}(t,t_0,s_0,s)H(s)ds \\ &= \int_{-\infty}^{\infty}\rho(t,t_0,s_0,s)\left(\left(G_1(t,s) + \frac{\Gamma(t,t_0)}{2}\frac{\partial\left(G_2(t,s)^2\right)}{\partial s}\right)H'(s) + \Gamma(t,t_0)G_2(t,s)H''(s)\right)ds \\ &= -\int_{-\infty}^{\infty}\frac{\partial}{\partial s}\left(\rho(t,t_0,s_0,s)\left(G_1(t,s) + \frac{\Gamma(t,t_0)}{2}\frac{\partial\left(G_2(t,s)^2\right)}{\partial s}\right)\right)H(s)ds \\ &\quad + \Gamma(t,t_0)\int_{-\infty}^{\infty}\frac{\partial^2}{\partial s^2}\left(G_2(t,s)^2\rho(t,s)\right)H(s)ds \end{aligned} \tag{47}$$

Since $H(s)$ is arbitrary, (47) implies

$$\frac{\partial\rho(t,t_0,s_0,s)}{\partial t} = -\frac{\partial}{\partial s}\left(\left(G_1(t,s) + \frac{\Gamma(t,t_0)}{2}\frac{\partial}{\partial s}\left(G_2(t,s)^2\right)\right)\rho(t,t_0,s_0,s)\right) + \Gamma(t,t_0)\frac{\partial^2}{\partial s^2}\left(G_2(t,s)^2\rho(t,t_0,s_0,s)\right) \tag{48}$$

which is equivalent to (32) which assumes differentiable noise $\zeta(t)$. Equation (48) expresses that the random draw causes a realization of the asset value $S(t)$ which leads to a probability density $\rho(t,t_0,s_0,s)$ which is equivalent for nondifferentiable and differentiable noise $\zeta(t)$. Hence, the partial differential equation in (48) for the probability density $\rho(t,t_0,s_0,s)$ can be solved numerically by applying random draws with corresponding averaging. See Appendix G for three examples.

## 6. Limitations, Opportunities, and Future Research

The use of differentiable noise $\zeta(t)$ may potentially introduce numerical stability issues. Future research should develop models to mitigate instability while preserving the benefits of differentiability. The model relies on accurately estimating $\tau$ and $\beta$. Advanced parameter estimation techniques, such as maximum likelihood estimation or Bayesian inference, could be integrated to enhance the robustness of the parameter estimates. For example, instead of the product form $\Pi(|t-t'|) \equiv \Pi_1(|t-t'|)\Pi_2(|t-t'|)$ in (6), alternative functional forms may be explored which may potentially better capture market dynamics according to empirical validation. Two examples are a power-law correlation function or a combination of exponential decays.

When developing the generalized Fokker–Planck equation in (32) to account for differentiable noise $\zeta(t)$, the definition $\Gamma(t, t_0) \equiv \int_{t_0}^{t} E(\xi(t)\xi(v))dv$ in (3) is used to represent the evolution of the probability density $\rho(t, t_0, s_0, s)$ for the asset value $S(t)$. Since the generalized Fokker–Planck equation is more complex than the traditional one, analytical solutions may be challenging to obtain. The presence of the term $\Gamma(t, t_0)$ in (32) complicates the direct integration and solution of the equation. Analogously, the complexity of the boundary conditions and initial values in real-world applications can be problematic. For example, the assumption that the density $\rho(t, t_0, s_0, s)$ is zero at the boundaries $s = -\infty$ and $s = \infty$ in (46) may not hold in practical scenarios. Some of these challenges may be addressed as follows:

1. Simplified numerical approaches or approximations may be developed that can handle the complexity of the generalized Fokker–Planck equation in (32). For instance, finite difference or spectral methods could be adapted to better accommodate the additional terms introduced by $\Gamma(t, t_0)$.
2. Stochastic simulation techniques, such as Monte Carlo methods, may be applied to approximate the solutions of the generalized Fokker–Planck equation in (32). These simulations may help understand the distributional properties of $S(t)$ without solving the equation analytically.
3. Regularization techniques may be utilized to stabilise the numerical solutions. For example, Tikhonov regularization or other smoothing methods could be applied to the generalized Fokker–Planck equation in (32) to handle the complexities introduced by the differentiable noise terms.
4. Hybrid models may be considered that blend differentiable and nondifferentiable noise processes. This could simplify the correlation structure while retaining the advantages of differentiable noise in modelling and forecasting.

Addressing the practical numerical implementation and parameter estimation challenges through hybrid approaches and advanced estimation techniques may enhance the model's applicability and performance.

## 7. Conclusion

This article assumes differentiable noise with nonzero correlation time and uses a generalized Novikov's [7] theorem for Gaussian noise to develop a Kolmogorov [8] backward equation for the probability density of an asset value as a function of time. An example of the model is to develop an alternative to the Black and Scholes [1] partial differential equation in finance, which plays a major role in the pricing of financial instruments, especially options and futures. Our partial differential equation is generalized to account for nonzero correlation time for the derivative of the noise with respect to time.

Coincidence exists between solutions applying the differentiable and nondifferentiable Gaussian noise when the Fisk–Stratonovich [13, 14] integrals are used for nondifferentiable noise. Realizations and solutions of the evolution of an asset value based on randomly drawing noise from probability distributions are developed, discussed, and analyzed. Mathematical models based on differentiable noise can improve forecasts.

The article contrasts with the literature which commonly applies the Ito [2] stochastic integral to develop the Kolmogorov [8] equation or Fokker–Planck equation [9, 10] from continuous nondifferentiable noise with zero correlation time.

## Appendix A. A Differentiable Noise Function as an Example

Let the noise derivative be

$$\xi(t) = \mathrm{Sin}(t + \theta) \tag{A.1}$$

where $t$ is time and $\theta$ is uniformly distributed, that is,

$$\rho_\theta(\theta) = \begin{cases} 1/2\pi, & \text{when} \quad 0 \le \theta \le 2\pi \\ 0, & \text{otherwise} \end{cases} \tag{A.2}$$

The probability $P(\mathrm{Sin}(t + \theta) \le \xi)$ that $\xi(t) = \mathrm{Sin}(t + \theta)$ is less than or equal to $\xi$ is given as

$$\begin{aligned} P(\mathrm{Sin}(t+\theta) \le \xi) &= 1 - \sum_i P(\mathrm{ArcSin}(\xi) + 2\pi i \le t + \theta \\ &\le \pi - \mathrm{ArcSin}(\xi) + 2\pi i) \\ &= 1 - \sum_i \int_{\mathrm{ArcSin}(\xi)-t+2\pi i}^{\pi-\mathrm{ArcSin}(\xi)-t+2\pi i} \rho_\theta(\theta)d\theta \end{aligned} \tag{A.3}$$

Differentiating the probability $P(\mathrm{Sin}(t + \theta) \le \xi)$ in (A.3) with respect to the noise derivative $\xi(t)$ gives the probability density

$$\begin{aligned} \rho_\xi(\xi) &= \frac{\partial}{\partial \xi} P(\mathrm{Sin}(t+\theta) \le \xi) = -\frac{\partial}{\partial \xi}\sum_i \int_{\mathrm{ArcSin}(\xi)-t+2\pi i}^{\pi-\mathrm{ArcSin}(\xi)-t+2\pi i} \rho_\theta(\theta)d\theta \\ &= \frac{1}{\left(1-\xi^2\right)^{1/2}} \sum_i [\rho_\theta(\pi - \mathrm{ArcSin}(\xi) - t + 2\pi i) \\ &\quad + \rho_\theta(\mathrm{ArcSin}(\xi) - t + 2\pi i)] = \frac{1}{\pi\left(1-\xi^2\right)^{1/2}}, \xi^2 \le 1 \end{aligned} \tag{A.4}$$

where

$$\int_{-1}^{1}\rho_\xi(\xi)d\xi=\int_{-1}^{1}\frac{1}{\pi\left(1-\xi^2\right)^{1/2}}d\xi=\frac{(\mathrm{ArcSin}(1)-\mathrm{ArcSin}(-1))}{\pi}=1$$
$$E(\Xi)=\int_{-1}^{1}\rho_\xi(\xi)\xi d\xi=\int_{-1}^{1}\frac{\xi}{\pi\left(1-\xi^2\right)^{1/2}}d\xi=0$$
$$E\left(\Xi^2\right)=\int_{-1}^{1}\rho_\xi(\xi)\xi^2 d\xi=\int_{-1}^{1}\frac{\xi^2}{\pi\left(1-\xi^2\right)^{1/2}}d\xi=\frac{1}{2} \tag{A.5}$$

To calculate correlation, inserting $t'=t+v$ into (A.5) gives

$$\begin{aligned}\xi\left(t'\right)&=\xi(t+v)=\mathrm{Sin}(t+v+\theta)\\&=\mathrm{Sin}(v)\mathrm{Cos}(t+\theta)+\mathrm{Cos}(v)\mathrm{Sin}(t+\theta)\\&=\mathrm{Sin}(v)\left(1-\xi(t)^2\right)^{1/2}+\mathrm{Cos}(v)\xi(t)\end{aligned} \tag{A.6}$$

which implies

$$\begin{aligned}&\rho_{\xi\xi'}\left(\xi\left(t'\right),\xi(t)\right)=\rho_\xi(\xi(t))\rho_{\xi'/\xi}\left(\xi\left(t'\right),\xi(t)\right)\\&=\frac{1}{\pi\left(1-\xi^2\right)^{1/2}}\delta\left(\xi\left(t'\right)-\mathrm{Sin}(v)\left(1-\xi(t)^2\right)^{1/2}-\mathrm{Cos}(v)\xi(t)\right)\end{aligned} \tag{A.7}$$

which gives the correlation

$$\begin{aligned}E\left(\xi\left(t'\right),\xi(t)\right)&=\int_{-1}^{1}\int_{-1}^{1}\xi\left(t'\right)\xi(t)\rho_{\xi\xi'}\left(\xi\left(t'\right),\xi(t)\right)d\xi\left(t'\right)d\xi(t)\\&=\int_{-1}^{1}\int_{-1}^{1}\frac{\delta\left(\xi\left(t'\right)-\mathrm{Sin}(v)\left(1-\xi^2\right)^{1/2}-\mathrm{Cos}(\mathrm{v})\xi(t)\right)\xi\left(t'\right)\xi(t)}{\pi\left(1-\xi^2\right)^{1/2}}d\xi\left(t'\right)d\xi(t)\\&=\int_{-1}^{1}\frac{\left(\mathrm{Sin}(v)\left(1-\xi^2\right)^{1/2}+\mathrm{Cos}(v)\xi\right)\xi}{\pi\left(1-\xi^2\right)^{1/2}}d\xi\\&=\frac{\mathrm{Sin}(v)}{\pi}\int_{-1}^{1}\xi d\xi+\mathrm{Cos}(v)\int_{-1}^{1}\frac{\xi^2}{\pi\left(1-\xi^2\right)^{1/2}}d\xi\\&=\mathrm{Cos}(v)E\left(\xi^2\right)=\frac{\mathrm{Cos}(v)}{2}=\frac{\mathrm{Cos}\left(\left|t-t'\right|\right)}{2}\end{aligned} \tag{A.8}$$

which implies

$$\begin{aligned}\Gamma(t,t_0)&\equiv\int_{t_0}^{t}E(\xi(t),\xi(v))dv=\frac{1}{2}\int_{t_0}^{t}\mathrm{Cos}(t-v)dv=\frac{1}{2}\int_{t_0-t}^{0}\mathrm{Cos}(z)dz=\frac{1}{2}\mathrm{Sin}(t-t_0)\\\widetilde{\Gamma}(t,t_0)&\equiv\int_{t_0}^{t}E(\xi(t_0),\xi(v))dv=\frac{1}{2}\int_{t_0}^{t}\mathrm{Cos}(t_0-v)dv=\frac{1}{2}\int_{0}^{t-t_0}\mathrm{Cos}(z)dz=\frac{1}{2}\mathrm{Sin}(t-t_0)\end{aligned} \tag{A.9}$$

Inserting the above equations into the variance of the noise $\zeta(t)$ at time $t$ implies

$$\begin{aligned}\Omega^{(2)}\left(t,t',t_0\right)&\equiv\int_{t_0}^{t'}\int_{t_0}^{t}E(\xi(u)\xi(v))dudv\\&=\frac{1}{2}\int_{t_0}^{t'}\int_{t_0}^{t}\mathrm{Cos}(u-v)dudv\\&=\frac{1}{2}\int_{t_0}^{t'}\int_{t_0-v}^{t-v}\mathrm{Cos}(z)dzdv\\&=\frac{1}{2}\int_{t_0}^{t'}(\mathrm{Sin}(t-v)-\mathrm{Sin}(t_0-v))dv\\&=\frac{1}{2}\left(-\mathrm{Cos}(t-t_0)-\mathrm{Cos}\left(t'-t_0\right)+1+\mathrm{Cos}\left(t-t'\right)\right)\end{aligned} \tag{A.10}$$

$$\frac{\partial^2\Omega^{(2)}\left(t,t',t_0\right)}{\partial t\partial t'}=\frac{1}{2}\mathrm{Cos}\left(t-t'\right)=E\left(\xi\left(t'\right),\xi(t)\right) \tag{A.11}$$

Inserting $t=t'$ into (A.10) and (A.11) implies

$$\begin{aligned}&\Omega^{(2)}(t,t,t_0)=1-\mathrm{Cos}(t-t_0)\\&\frac{\partial\Omega^{(2)}(t,t,t_0)}{\partial t}=\mathrm{Sin}(t-t_0)=2\Gamma(t,t_0)\end{aligned} \tag{A.12}$$

$$\partial\Omega^{(2)}\frac{(t,t,t_0)}{\partial t_0}=-\mathrm{Sin}(t-t_0)=-2\Gamma(t,t_0)=-2\widetilde{\Gamma}(t,t_0) \tag{A.13}$$

## Appendix B. Random Variable and Stochastic Process

Let $X$ be a random variable, where $X$ is a function

$$X:\Omega\longrightarrow R^n \tag{B.1}$$

where $\Omega$ is the set of possible outcomes and $X$ assigns a number in $R^n$ to each element in $\Omega$.

As an example, consider two coins with an $M$ or $K$ for each coin. The set of possible outcomes in tossing both coins once is $\Omega=\{\mathrm{MK},\mathrm{MM},\mathrm{KK},\mathrm{KM}\}$. $X$ relates a real number to each of the elements of this set, say for example

$$\begin{aligned}X(\mathrm{MK})&=1,\\X(\mathrm{MM})&=2,\\X(\mathrm{KK})&=3,\\X(\mathrm{KM})&=4\end{aligned} \tag{B.2}$$

Let $\Lambda$ be the sigma-algebra of subsets of $\Omega$. Thus, $\Lambda$ has for the case $\Omega=\{\mathrm{MK},\mathrm{MM},\mathrm{KK},\mathrm{KM}\}$ 15 elements, to read

$$\Lambda = \{\{\mathrm{MK}\}, \{\mathrm{MM}\}, \{\mathrm{KK}\}, \{\mathrm{KM}\}, \{\mathrm{MK, MM}\}, \{\mathrm{MK, KK}\}, \{\mathrm{MK, KM}\}, \{\mathrm{MM, KK}\}, \{\mathrm{MM, KM}\}, \{\mathrm{KK, KM}\}, \{\mathrm{MK, MM, KK}\}, \{\mathrm{MK, KK, KM}\}, \{\mathrm{MM, KK, KM}\}, \{MK, MM, KK, KM\}, \emptyset\} \tag{B.3}$$

Consider the probability $P$ defined as $P : \Lambda \longrightarrow [0, 1]$. The probability $P$ is a function that relates a number in the interval $[0, 1]$ to each member in $\Lambda$, where $P(\emptyset) = 0$. Thus, $P$ is a measure on $\Omega$. We may set for the case above

$$\begin{aligned} &P(\{\mathrm{MK}\}) = P(\{\mathrm{MM}\}) = P(\{\mathrm{KK}\}) = P(\{\mathrm{KM}\}) = \frac{1}{4}, \\ &P(\{\mathrm{MK, MM}\}) = P(\{\mathrm{MK, KK}\}) = P(\{\mathrm{MK, KM}\}) \\ &\quad = P(\{\mathrm{MM, KK}\}) = P(\{\mathrm{MM, KM}\}) \\ &\quad = P(\{\mathrm{KK, KM}) = \frac{1}{2}, \\ &P(\{MK, \mathrm{MM}, KK\}) = P(\{\mathrm{MK, KK, KM}\}) \\ &\quad = P(\{\mathrm{MM, KK, KM}\}) = \frac{3}{4}, \\ &P(\{\mathrm{MK, MM, KK, KM}\}) = 1, P(\emptyset) = 0 \end{aligned} \tag{B.4}$$

Generally, we let $B$ denote a Borel set in $R^n$, such that

$$X^{-1} : B \longrightarrow \Lambda \tag{B.5}$$

which also can be expressed as $X^{-1}(B) \in \Lambda$. For the example above, $P(\{MK\})$ equals the probability that $X = 1$, that is, $P(X^{-1}(1)) = P\{MK\}) = 1/2$.

A stochastic process is a parametrized collection of random variables

$$\{X_t\}_{t\in T} \tag{B.6}$$

where $T$ is usually the half line $[0, \infty)$, an interval, or the nonnegative integers.

The most important restriction for martingales is that $\{X_t\}_{t\in T}$ is a martingale if

$$E\left(\frac{X_s}{\{x_t\}}\right) = X_s, s \geq t \tag{B.7}$$

where $\{x_t\}$ is a nondecreasing sequence of $X_t$.

## Appendix C. Examples for Section 2 When $c = 1/2$ and $c = 1$

Inserting $c = 1/2$ into (12) gives

$$\begin{aligned} \Omega^{(2)}(t, t_0) &= \sigma^2 \int_{t_0}^{t} \left[\Gamma_f(1, 0) - \Gamma_f\left(1, \frac{\nu - t_0}{\tau}\right)\right] d\nu \\ &= \sigma^2 \int_{t_0}^{t} \left[1 - Exp\left(\frac{(t_0 - \nu)}{\tau}\right)\right] d\nu \\ &= \sigma^2 \left[t - t_0 - \tau + \tau \mathrm{Exp}\left(-\frac{t - t_0}{\tau}\right)\right] \end{aligned} \tag{C.1}$$

Inserting $c = 1$ into (12) gives

$$\begin{aligned} \Omega^{(2)}(t, t_0) &= \sigma^2 \frac{\tau}{\beta} \int_{t_0}^{t} \left[\Gamma_f(2, 0) - \Gamma_f\left(2, \frac{\nu - t_0}{\tau}\right)\right] d\nu \\ &= \sigma^2 \frac{\tau}{\beta} \int_{t_0}^{t} \left[1 - \Gamma_f\left(2, \frac{\nu - t_0}{\tau}\right)\right] d\nu \\ &= \sigma^2 \frac{\tau}{\beta} \left[t - t_0 - 2\tau - \mathrm{Exp}\left(-\frac{t - t_0}{\tau}\right)(-(t - t_0) - 2\tau)\right] \end{aligned} \tag{C.2}$$

Hence, $\partial\Omega^{(2)}/\partial t_0 = -2\widetilde{\Gamma}(t, t_0) = \;=\; -\partial\Omega^{(2)}/\partial t$, which applies generally.

Inserting (12) when $c = 1/2$ into (14) gives

$$\Omega^{(2)}(t, t_0) = \begin{cases} \sigma^2(t - t_0 - \tau), & \dfrac{(t - t_0)}{\tau} \gg 1 \\ \sigma^2\left(\dfrac{(t - t_0)^2}{2\tau}\right), & \dfrac{(t - t_0)}{\tau} \ll 1 \end{cases} \tag{C.3}$$

If the noise derivative $\xi(t)$ is Gaussian distributed, taking the limit $\tau \longrightarrow 0$ in (C.3) gives Brownian motion, that is,

$$\begin{aligned} &\mathrm{Lim}_{\tau\longrightarrow 0} \frac{\Pi_1(|u|)}{2\tau} = \delta(u), \\ &\Omega^{(2)}(t, t_0) \equiv \int_{t_0}^{t}\int_{t_0}^{t} E(\xi(u)\xi(\nu)) du d\nu \\ &\quad = \sigma^2 \int_{t_0}^{t}\int_{t_0}^{t} \delta(u - \nu) du d\nu \end{aligned} \tag{C.4}$$

Inserting (C.2) when $c = 1$ into (14) gives

$$\Omega^{(2)}(t, t_0) = \begin{cases} \sigma^2 \dfrac{\tau}{\beta}(t - t_0 - 2\tau), & \dfrac{t - t_0}{\tau} \gg 1 \\ \sigma^2 \dfrac{\tau}{\beta}\left(\dfrac{(t - t_0)^3}{(6\tau^2)}\right), & \dfrac{t - t_0}{\tau} \ll 1 \end{cases} \tag{C.5}$$

Assuming $t \geq t'$ and inserting $c = 1/2$ into (15) implies

$$\Omega^{(2)}\left(t, t', t_0\right) = \frac{\sigma^2}{2}\frac{\tau^{2c-1}}{\beta^{2c-1}}\int_{t_0}^{t'}\left[\Gamma_f(2c, 0) - \Gamma_f\left(2c, \frac{\nu - t_0}{\tau}\right) + \Gamma_f(2c, 0) - \Gamma_f\left(2c, \frac{t-\nu}{\tau}\right)\right]d\nu$$
$$= \frac{\sigma^2}{2}\int_{t_0}^{t'}\left[1 - \mathrm{Exp}\left(\frac{(t_0 - \nu)}{\tau}\right) + 1 - \mathrm{Exp}\left(\frac{(-t+\nu)}{\tau}\right)\right]d\nu$$
$$= \frac{\sigma^2}{2}\left[2t' - 2t_0 + \tau\left(\mathrm{Exp}\left(\frac{t_0 - t'}{\tau}\right) + \mathrm{Exp}\left(\frac{t_0 - t}{\tau}\right)\right) - \tau\left(1 + \mathrm{Exp}\left(\frac{t'-t}{\tau}\right)\right)\right], t \geq t' \geq t_0 \quad \text{(C.6)}$$

Assuming $t' \geq t$ and inserting $c = 1/2$ into (15) implies

$$\Omega^{(2)}\left(t, t', t_0\right) = \frac{\sigma^2}{2}\left[t + t' - \left|t - t'\right| - 2t_0 + \tau\left(\mathrm{Exp}\left(\frac{t_0 - t'}{\tau}\right) + Exp\left(\frac{t_0 - t}{\tau}\right)\right) - \tau\left(1 + Exp\left(-\frac{\left|t - t'\right)\right|}{\tau}\right)\right)\right] \quad \text{(C.7)}$$

Inserting $t = t'$ into (C.6) and (C.7) gives $\Omega^{(2)}(t, t, t_0) = \sigma^2[t - t_0 + \tau(\mathrm{Exp}(-(t - t_0)/\tau) - 1)]$, which confirms (12). Equations (C.6) and (C.7) can also be expressed as

$$\Omega^{(2)}\left(t, t', t_0\right) = \begin{cases} \frac{\sigma^2}{2}\left(t + t' - \left|t - t'\right| - 2t_0 - \tau\right), \frac{t - t_0}{\tau} \gg 1, \frac{t' - t_0}{\tau} \gg 1, \frac{\left|t - t'\right|}{\tau} \gg 1 \\ \frac{\sigma^2}{2}\left(\frac{(t - t_0)^2}{2\tau} + \frac{\left(t' - t_0\right)^2}{2\tau} + \frac{\left|t - t'\right|^2}{2\tau}\right), \frac{t - t_0}{\tau} \ll 1, \frac{t' - t_0}{\tau} \ll 1, \frac{\left|t - t'\right|}{\tau} \ll 1 \end{cases} \quad \text{(C.8)}$$

Equation (C.8) implies $\mathrm{Lim}_{\tau \longrightarrow 0}\Omega^{(2)}(t, t', t_0) = \sigma^2[(t - t_0) + (t' - t_0) - |t - t'|]/2$, which is Brownian motion if the noise derivative $\xi(t)$ is Gaussian.

Assuming $t \geq t'$ and inserting $c = 1$ into (15) imply

$$\Omega^{(2)}\left(t, t', t_0\right) = \frac{\sigma^2}{2}\frac{\tau}{\beta}\int_{t_0}^{t'}\left[\Gamma_f(2, 0) - \Gamma_f\left(2, \frac{\nu - t_0}{\tau}\right) + \Gamma_f(2, 0) - \Gamma_f\left(2, \frac{t - \nu}{\tau}\right)\right]d\nu$$
$$= \frac{\sigma^2}{2}\frac{\tau}{\beta}\int_{t_0}^{t'}\left[1 - \Gamma_f\left(2, \frac{\nu - t_0}{\tau}\right) + 1 - \Gamma_f\left(2, \frac{t - \nu}{\tau}\right)\right]d\nu$$
$$= \frac{\sigma^2}{2}\frac{\tau}{\beta}\left(-2\tau + t' + \mathrm{Exp}\left(\frac{t_0 - t'}{\tau}\right)\left(2\tau + t' - t_0\right) - t_0\right) + t' - t_0 + Exp\left(-\frac{t}{\tau}\right)\left(\mathrm{Exp}\left(\frac{t'}{\tau}\right)\left(-t - 2\tau + t'\right) + \mathrm{Exp}\left(\frac{t_0}{\tau}\right)(t + 2\tau - t_0)\right)$$
$$= \frac{\sigma^2}{2}\frac{\tau}{\beta}\left(2t' - 2t_0 - 2\tau + Exp\left(\frac{t_0 - t'}{\tau}\right)\left(t' - t_0 + 2\tau\right) + \mathrm{Exp}\left(\frac{t_0 - t}{\tau}\right)(t - t_0 + 2\tau) + \mathrm{Exp}\left(\frac{t' - t}{\tau}\right)\left(t' - t - 2\tau\right)\right),$$
$$t \geq t' \geq t_0 \quad \text{(C.9)}$$

Assuming $t' \geq t$ and inserting $c = 1$ into (15) imply

$$\Omega^{(2)}\left(t, t', t_0\right) = \frac{\sigma^2}{2}\frac{\tau}{\beta}\left(t + t' - \left|t - t'\right| - 2t_0 - 2\tau + \mathrm{Exp}\left(\frac{t_0 - t'}{\tau}\right) \cdot \left(t' - t_0 + 2\tau\right) + \mathrm{Exp}\left(\frac{t_0 - t}{\tau}\right)(t - t_0 + 2\tau) + \mathrm{Exp}\left(-\frac{\left|t - t'\right|}{\tau}\right)\left(-\left|t - t'\right| - 2\tau\right)\right) \quad \text{(C.10)}$$

Inserting $t = t'$ into (C.9) and (C.10) gives $\Omega^{(2)}(t, t, t_0) = \sigma^2(\tau/\beta)[t - t_0 - 2\tau + \mathrm{Exp}(-(t - t_0)/\tau)((t - t_0) + 2\tau)]$, which confirms (C.2).

## Appendix D. Kolmogorov [8] Forward Equation and Novikov's [7] Identity

Consider the differential equation in (1), that is,

$$\dot{S}(t) = G_1(t, S(t)) + G_2(t, S(t))\xi(t) \quad \text{(D.1)}$$

where $G_2(t, S(t)) = f(S(t))g(t)$. The function $g(t)$ can be absorbed into the noise derivative $\xi(t)$ to give the correlation strength $\sigma^2$ of the noise dependent on time $t$. Thus, we set that $G_2 = G_2(S(t)) = G_2(S)$ is independent of time $t$.

We introduce

$$\Phi_s(S) \equiv \int_{-\infty}^{\infty}\frac{1}{G_2(S)}dS \quad \text{(D.2)}$$

and assume invertibility such that $S = S_\Phi(\Phi)$, where $\Phi(t) \equiv \Phi_s(S(t))$. Differentiating (D.2) with respect to time $t$ gives

$$\dot{\Phi}(t) = \frac{\partial\Phi_s}{\partial S}\dot{S}(t) = \frac{\dot{S}(t)}{G_2(S)} = \frac{G_1(t, S)}{G_2(S)} + \xi(t) = \Omega(t, \Phi(t)) + \xi(t) \quad \text{(D.3)}$$

where

$$\Omega(t, \Phi) \equiv \frac{G_1(t, S_\Phi(\Phi))}{G_2(S_\Phi(\Phi))} \quad \text{(D.4)}$$

The Fokker–Planck equation [9, 10] for $\Phi(t)$ applies Novikov's [7] identity

$$\dot{\rho}_\phi(t, \phi) = -\frac{\partial}{\partial\phi}\left(\rho_\phi(t, \phi)\Omega(t, \phi)\right) + \Gamma\frac{\partial^2}{\partial\phi^2}\left(\rho_\phi(t, \phi)\right) \quad \text{(D.5)}$$

where $\rho_\phi(t, \phi)$ is the probability density for $\Phi(t)$.

Let $\rho(t, t_0, s_0, s)$ be the density of the asset value $S(t)$ as defined in Section 3. The probability $P(\Phi \leq \phi)$ thus becomes

$$P(\Phi \leq \phi) = \int_{-\infty}^{\phi}\rho_\phi(t, u)du = P(S \leq S_\Phi(\phi)) = \int_{-\infty}^{S_\Phi(\phi)}\rho(t, t_0, s_0, u)du \quad \text{(D.6)}$$

Differentiating (D.6) with respect to $\Phi(t)$ gives

$$\frac{\partial}{\partial\phi}P(\Phi\le\phi)=\rho_\phi(t,\phi)=\rho(t,t_0,s_0,S_\Phi(\phi))S'_\Phi(\phi)\\=\rho(t,t_0,s_0,S_\Phi(\phi))G_2(t,S_\Phi(\phi)) \tag{D.7}$$

which implies

$$\frac{\rho_\phi\left(t,S_\Phi^{-1}(s)\right)}{S'_\Phi(\phi)}=\rho(t,t_0,s_0,s) \tag{D.8}$$

Assuming an arbitrary function $H(S)$ of the asset value $S$ and applying partial integration and that $\rho(t,t_0,s_0,s)$ is zero at the boundaries $s=-\infty$ and $s=\infty$ imply

$$\begin{aligned}\frac{d}{dt}E(H(S))&=\int_{-\infty}^{\infty}\dot\rho(t,t_0,s_0,s)H(s)ds\\&=\int_{-\infty}^{\infty}\dot\rho_\phi(t,\phi)H(S_\Phi(\phi))d\phi\\&=\int_{-\infty}^{\infty}H(S_\Phi(\phi))\left[-\frac{\partial}{\partial\phi}\left(\rho_\phi(t,\phi)\Omega\right)+\Gamma\frac{\partial^2}{\partial\phi^2}\left(\rho_\phi(t,\phi)\right)\right]d\phi\\&=\int_{-\infty}^{\infty}\rho_\phi(t,\phi)H'(S_\Phi(\phi))S'_\Phi(\phi)\frac{G_1(t,S_\Phi(\phi))}{G_2(S_\Phi(\phi))}d\phi\\&\quad+\Gamma\int_{-\infty}^{\infty}\rho_\phi(t,\phi)\frac{\partial^2(H(S_\Phi(\phi)))}{\partial\phi^2}d\phi\end{aligned} \tag{D.9}$$

where

$$\begin{aligned}S'_\Phi(\Phi)&=G_2(S_\Phi(\Phi))\\S''_\Phi(\Phi)&=G'_2(S_\Phi(\Phi))S'_\Phi(\Phi)=G'_2(S_\Phi(\Phi))G_2(S_\Phi(\Phi))\\&=\frac{1}{2}\frac{\partial}{\partial S_\Phi}\left(G_2(S_\Phi(\Phi))^2\right)\end{aligned} \tag{D.10}$$

$$\begin{aligned}S'_\Phi(\phi)\frac{G_1(S_\Phi(\phi))}{G_2(S_\Phi(\phi))}&=G_1(S_\Phi(\phi))\frac{\partial^2(H(S_\Phi(\phi)))}{\partial\phi^2}\\&=\frac{\partial}{\partial\phi}\left(H'(S_\Phi(\phi))S'_\Phi(\phi)\right)\\&=S''_\Phi(\phi)H'(S_\Phi(\phi))+S'_\Phi(\phi)^2H''(S_\Phi(\phi))\\&=\frac{1}{2}\frac{\partial}{\partial S_\Phi}\left(G_2(S_\Phi(\phi))^2\right)H'(S_\Phi(\phi))\\&\quad+G_2(S_\Phi(\phi))^2H''(S_\Phi(\phi))\end{aligned} \tag{D.11}$$

Hence,

$$\begin{aligned}&\int_{-\infty}^{\infty}\dot\rho(t,t_0,s_0,s)H(s)ds\\&=\int_{-\infty}^{\infty}\rho_\phi(t,\phi)G_1(t,S_\Phi(\phi))H'(S_\Phi(\phi))d\phi\\&\quad+\Gamma\int_{-\infty}^{\infty}\rho_\phi(t,\phi)\frac{1}{2}\frac{\partial}{\partial S_\Phi}\left(G_2(t,S_\Phi(\phi))^2\right)H'(S_\Phi(\phi))d\phi\\&\quad+\Gamma\int_{-\infty}^{\infty}\rho_\phi(t,\phi)G_2(t,S_\Phi(\phi))^2H''(S_\Phi(\phi))d\phi\\&=\int_{-\infty}^{\infty}\rho(t,t_0,s_0,s)G_1(t,s)H'(s)ds\\&\quad+\frac{\Gamma}{2}\int_{-\infty}^{\infty}\rho(t,t_0,s_0,s)\frac{\partial\left(G_2(t,s)^2\right)}{\partial s}H'(s)ds\\&\quad+\Gamma\int_{-\infty}^{\infty}\rho(t,t_0,s_0,s)G_2(t,s)^2H''(s)ds\end{aligned} \tag{D.12}$$

Applying partial integration and that the density $\rho(t,t_0,s_0,s)$ is zero at the boundaries $s=-\infty$ and $s=\infty$ gives

$$\begin{aligned}&\int_{-\infty}^{\infty}H(s)\dot\rho(t,t_0,s_0,s)ds\\&=\int_{-\infty}^{\infty}H(s)\left[-\frac{\partial}{\partial s}\left(\left(G_1(t,s)+\frac{\Gamma}{2}\frac{\partial}{\partial s}\left(G_2(t,s)^2\right)\right)\rho(t,t_0,s_0,s)\right)\right.\\&\quad\left.+\Gamma\frac{\partial^2}{\partial s^2}\left(G_2(t,s)^2\rho(t,t_0,s_0,s)\right)\right]ds\end{aligned} \tag{D.13}$$

Thus, since $H(S)$ is arbitrary, differentiating the density with respect to time $t$ gives

$$\begin{aligned}\frac{\partial\rho(t,t_0,s_0,s)}{\partial t}&=-\frac{\partial}{\partial s}\left(\left(G_1(t,s)+\frac{\Gamma}{2}\frac{\partial}{\partial s}\left(G_2(t,s)^2\right)\right)\rho(t,t_0,s_0,s)\right)\\&\quad+\Gamma\frac{\partial^2}{\partial s^2}\left(G_2(t,s)^2\rho(t,t_0,s_0,s)\right)\end{aligned} \tag{D.14}$$

Hence, Novikov's [7] generalized identity is proved by inverse calculation.

## Appendix E. The Kolmogorov [8] Backward Equation

$$\begin{aligned}&\frac{P(x,t/x_0,t_0-h)-P(x,t/x_0,t_0)}{h}\\&=\frac{1}{h}\int_{-\infty}^{\infty}\left(P\left(x,\frac{t}{y},t_0\right)P\left(y,\frac{t_0}{x_0},t_0-h\right)-P\left(x,\frac{t}{x_0},t_0\right)\right.\\&\quad\left.\cdot P\left(y,\frac{t_0}{y},t_0-h\right)\right)dy=\frac{1}{h}\int_{-\infty}^{\infty}\left(P\left(x,\frac{t}{y},t_0\right)\right.\\&\quad\left.-P\left(x,\frac{t}{x_0},t_0\right)\right)P\left(y,\frac{t_0}{x_0},t_0-h\right)dy\end{aligned} \tag{E.1}$$

Moreover,

$$P\left(x, \frac{t}{y}, t_0\right) - P\left(x, \frac{t}{x_0}, t_0\right) = \frac{\partial P(x, t/x_0, t_0)}{\partial x_0}(y - x_0) + \frac{1}{2}\frac{\partial^2 P(x, t/x_0, t_0)}{\partial x_0^2}(y - x_0)^2 \tag{E.2}$$

Thus

$$-\frac{\partial P(x, t/x_0, t_0 - h)}{\partial t_0} = \frac{\partial P(x, t/x_0, t_0)}{\partial x_0}\frac{1}{h}\int_{-\infty}^{\infty}(y - x_0)P\left(y, \frac{t_0}{x_0}, t_0 - h\right)dy + \frac{1}{2}\frac{\partial^2 P(x, t/x_0, t_0)}{\partial x_0^2}\frac{1}{h}\int_{-\infty}^{\infty}(y - x_0)^2 P\left(y, \frac{t_0}{x_0}, t_0 - h\right)dy \tag{E.3}$$

where $G_1(t_0, x_0) = (1/h)\int_{-\infty}^{\infty}(y - x_0)P(y, (t_0/x_0), t_0 - h)dy$ and $G_2(t_0, x_0)^2\sigma^2 = (1/h)\int_{-\infty}^{\infty}(y - x_0)^2 P(y, (t_0/x_0), t_0 - h)dy$.

## Appendix F. Riemann–Stieltjes, Ito, and Fisk–Stratonovich [12, 13] Integrals

Let $S(t)$ be given by the first-order ordinary Langevin [11] differential equation defined in (1), that is,

$$\dot{S}(t) = G_1(t, S(t)) + G_2(t, S(t))\xi(t) \tag{F.1}$$

which has the solution

$$S(t) = S(t_0) + \int_{t_0}^{t} G_1(u, S(u))du + \int_{t_0}^{t} G_2(u, S(u))\dot{\zeta}(u)du \tag{F.2}$$

where $\xi(t) \equiv \dot{\zeta}(t)$ is the noise derivative and the noise since $\zeta(t)$ is differentiable with respect to time $t$. The last term on the right-hand side in (F.2) is a traditional Riemann–Stieltjes integral, where

$$\begin{aligned}\int_{t_0}^{t} G_2(u, S(u))\xi(u)du &= \int_{t_0}^{t} G_2(u, S(u))\dot{\zeta}(u)du \\ &= \int_{t_0}^{t} G_2(u, S(u))d\zeta(u) \\ &= Lim_{\Delta \longrightarrow 0}\sum_{j=1}^{N-1} G_2(t_j, S(t_j))(\zeta(t_{j+1}) - \zeta(t_j)) \\ &= Lim_{\Delta \longrightarrow 0}\sum_{j=1}^{N-1} G_2(t_j, S(t_j))\xi(t_j)(t_{j+1} - t_j)\end{aligned} \tag{F.3}$$

where $t_0 = t_1 < t_2 < t_3 \cdots t_N = t, \Delta = \mathrm{Max}(t_{j+1} - t_j)$.

Assume that the noise derivative $\xi(t)$ is singular. It exists only in some generalized sense, that is, as the distributional derivative of some nondifferentiable continuous function, for instance, the Wiener noise/process (Brownian motion). Other processes are also possible, for example, the Poisson process or fractional Brownian motion. Then, $\zeta(t_{j+1}) - \zeta(t_j) \neq \xi(t_j)(t_{j+1} - t_j)$.

Applying the Wiener noise $W$ as an example of nondifferentiable noise, we may set

$$\int_{t_0}^{t} G_2(u, S(u))\xi(u)du = \int_{t_0}^{t} G_2(u, S(u))dW \tag{F.4}$$

The Ito [2] integral now implies

$$I\int_{t_0}^{t} G_2(u, S(u))dW = E\left(Lim_{\Delta \longrightarrow 0}\sum_{j=1}^{N-1} G_2(t_j, S(t_j))(W(t_{j+1}) - W(t_j))\right) \tag{F.5}$$

For the Wiener noise,

$$E(W(t_j)W(t_{j+1})) = \mathrm{Min}(t_j, t_{j+1}) = t_j, E(W(t_j)W(t_j)) = t_j \tag{F.6}$$

Thus

$$E\left(\sum_{j=1}^{N-1} W(t_j)(W(t_{j+1}) - W(t_j))\right) = 0 \tag{F.7}$$

The Fisk–Stratonovich [12, 13] integrals imply that

$$S\int_{t_0}^{t} G_2(u, S(u))dW \longrightarrow E\left(Lim_{\Delta \longrightarrow 0}\sum_{j=1}^{N-1} G_2\left(\frac{t_{j+1} + t_j}{2}, \frac{S(t_{j+1}) + S(t_j)}{2}\right)(W(t_{j+1}) - W(t_j))\right) \tag{F.8}$$

Thus,

$$\begin{aligned}&E\left(\sum_{j=1}^{N-1}\frac{1}{2}(W(t_{j+1}) + W(t_j))(W(t_{j+1}) - W(t_j))\right) \\ &= E\left(\sum_{j=1}^{N-1}\frac{1}{2}(W(t_{j+1})W(t_{j+1}) - W(t_{j+1})W(t_j) + W(t_j)W(t_{j+1}) - W(t_j)W(t_j))\right) \\ &= \sum_{j=1}^{N-1}\frac{1}{2}(t_{j+1} - t_j + t_j - t_j) = \frac{1}{2}(t_N - t_1) \\ &= E\left(\frac{1}{2}W(t_N)^2\right) - E\left(\frac{1}{2}W(t_1)^2\right)\end{aligned} \tag{F.9}$$

## Appendix G. Three Explicit Calculations as Examples

**Example 1.** Let

$$\dot{S}(t) = w(t) + \xi(t), G_1 = w, G_2 = 1 \tag{G.1}$$

The solution is

$$S(t,t_0,s_0)=s_0+\int_{t_0}^{t}w(u)du+\int_{t_0}^{t}\xi(u)du \tag{G.2}$$

The expectation is

$$E(S)=s_0+\int_{t_0}^{t}w(u)du+\Omega^1(t,t_0)=s_0+\int_{t_0}^{t}w(u)du$$

$$E(S^2)=\left(s_0+\int_{t_0}^{t}w(u)du\right)^2+\Omega^2(t,t_0)$$

$$\begin{aligned}E(S^n)&=E\left(\left(s_0+\int_{t_0}^{t}w(u)du+\int_{t_0}^{t}\xi(u)du\right)^n\right)\\&=\sum_{k=0}^{n}\binom{n}{k}\left(s_0+\int_{t_0}^{t}w(u)du\right)^k E\left[\left(\int_{t_0}^{t}\xi(u)du\right)^{n-k}\right]\end{aligned} \tag{G.3}$$

Thus,

$$\frac{\partial E(S)}{\partial t_0}=-w(t_0),\ \frac{\partial E(s)}{\partial s_0}=1,\ \frac{\partial^2E(S)}{\partial s_0^2}=0$$

$$\frac{\partial E(S^2)}{\partial \mathrm{t}_0}=2\left(s_0+\int_{\mathrm{t}_0}^{\mathrm{t}}w(u)du\right)(-w(t_0))+D_2\Omega^2(t,t_0)$$

$$\frac{\partial E(S^2)}{\partial s_0}=2\left(s_0+\int_{t_0}^{t}w(u)du\right),\ \frac{\partial E(S^2)}{\partial s_0^2}=2 \tag{G.4}$$

Hence,

$$\frac{\partial E(S)}{\partial t_0}+w(t_0)\frac{\partial E(S)}{\partial \mathrm{s}_0}=0$$

$$\frac{\partial E(S^2)}{\partial t_0}+w(t_0)\frac{\partial E(S^2)}{\partial s_0}+\widetilde{\Gamma}(t,t_0)\frac{\partial^2E(S^2)}{\partial {s_0}^2}=0 \tag{G.5}$$

**Example 2.** Let

$$\dot{S}(t)=bS(t)+\xi(t),\ G_1=bS(t),\ G_2=1 \tag{G.6}$$

The solution is

$$S(t,t_0,s_0)=s_0Exp(b(t-t_0))+\mathrm{Exp}(bt)\int_{t_0}^{t}\mathrm{Exp}(-bu)\xi(u)du \tag{G.7}$$

The expectation is

$$E(S)=s_0\mathrm{Exp}(b(t-t_0))$$

$$E(S^2)=s_0^2\mathrm{Exp}(2b(t-t_0))+\mathrm{Exp}(2bt)E\left(\left[\int_{t_0}^{t}\mathrm{Exp}(-bv)\xi(v)dv\right]^2\right) \tag{G.8}$$

Thus

$$\frac{\partial E(S)}{\partial t_0}=s_0\mathrm{Exp}(b(t-t_0))(-b),\ \frac{\partial E(S)}{\partial s_0}=\mathrm{Exp}(b(t-t_0)),\ \frac{\partial^2E(S)}{\partial s_0^2}=0 \tag{G.9}$$

$$\begin{aligned}\frac{\partial E(S^2)}{\partial t_0}&=s_0^2\mathrm{Exp}(2b(t-t_0))(-2b)-2\mathrm{Exp}(2bt)\\&\quad\cdot E\left(\int_{t_0}^{t}\mathrm{Exp}(-bv)\xi(v)\mathrm{Exp}(-bt_0)\xi(t_0)dv\right)\\&=s_0^2\mathrm{Exp}(2b(t-t_0))(-2b)-2\mathrm{Exp}(2bt-bt_0)\\&\quad\cdot\left(\int_{t_0}^{t}E(\mathrm{Exp}(-bv)\xi(v)\xi(t_0))dv\right)\end{aligned}$$

$$\frac{\partial E(S^2)}{\partial s_0}=2s_0\mathrm{Exp}(2b(t-t_0)),\ \frac{\partial^2E(S^2)}{\partial {s_0}^2}=2Exp(2b(t-t_0)) \tag{G.10}$$

Hence,

$$\frac{\partial E(S)}{\partial t_0}+bs_0\frac{\partial E(S)}{\partial s_0}=0 \tag{G.11}$$

$$\begin{aligned}&\frac{\partial E(S^2)}{\partial t_0}+bs_0\frac{\partial E(S^2)}{\partial s_0}+\widetilde{\Gamma}(t,t_0)\frac{\partial^2E(S^2)}{\partial {s_0}^2}\\&=s_0^2\mathrm{Exp}(2b(t-t_0))(-2b)-2\mathrm{Exp}(2bt-bt_0)\\&\quad\cdot\left(\int_{t_0}^{t}E(\mathrm{Exp}(-bv)\xi(v)\xi(t_0))dv\right)+2bs_0^2\mathrm{Exp}(2b(t-t_0))\\&\quad+2\mathrm{Exp}(2b(t-t_0))\int_{t_0}^{t}E(\xi(t_0)\xi(v))dv=0\end{aligned} \tag{G.12}$$

$$\begin{aligned}\int_{t_0}^{t}E(\xi(t_0)\mathrm{Exp}(-bv)\xi(v))dv&=\int_{t_0}^{t}\mathrm{Exp}(-bv)E(\xi(t_0)\xi(v))dv\\&=\int_{t_0}^{t}\mathrm{Exp}(-bt_0)E(\xi(t_0)\xi(v))dv\end{aligned} \tag{G.13}$$

**Example 3.** Let

$$\dot{S}(t)=w(t)S(t)+S(t)\xi(t),\ G_1=wS,\ G_2=S \tag{G.14}$$

The solution is

$$S(t,t_0,s_0)=s_0\mathrm{Exp}\left(\int_{t_0}^{t}w(u)du+\int_{t_0}^{t}\xi(u)du\right) \tag{G.15}$$

The expectation is

$$E(S) = s_0 \mathrm{Exp}\left(\int_{t_0}^{t} w(u)du\right) E\left(\mathrm{Exp}\left(\int_{t_0}^{t} \xi(u)du\right)\right)$$
$$E\left(S^2\right) = s_0^2 \mathrm{Exp}\left(2\int_{t_0}^{t} w(u)du\right) E\left(\mathrm{Exp}\left(\int_{t_0}^{t} 2\xi(u)du\right)\right) \tag{G.16}$$

Now,

$$\begin{aligned} E\left(\mathrm{Exp}\left(\int_{t_0}^{t} \xi(u)du\right)\right) &= E\left(1 + \left(\int_{t_0}^{t} \xi(u)du\right) + \frac{1}{2!}\left(\int_{t_0}^{t} \xi(u)du\right)^2 + \frac{1}{3!}\left(\int_{t_0}^{t} \xi(u)du\right)^4 + \cdots\right) \\ &= 1 + \Omega^1(t, t_0) + \frac{1}{2!}\Omega^2(t, t_0) + \frac{1}{3!}\Omega^3(t, t_0) + \frac{1}{4!}\Omega^4(t, t_0) + \cdots \end{aligned} \tag{G.17}$$

Assuming Gaussian noise $\zeta(t)$ in (G.17) implies

$$\begin{aligned} E\left(\mathrm{Exp}\left(\int_{t_0}^{t} \xi(u)du\right)\right) &= 1 + \sum_{n=1} \frac{1}{n!}\Omega^n(t, t_0) \\ &= 1 + \sum_{n=1} \frac{1}{(2n)!}\Omega^{2n}(t, t_0) \\ &= 1 + \sum_{n=1} \frac{1}{(2n)!}\frac{(2n)!}{n!}\left(\frac{\Omega^2(t, t_0)}{2}\right)^n \\ &= 1 + \sum_{n=1} \frac{1}{n!}\left(\frac{\Omega^2(t, t_0)}{2}\right)^n \\ &= \mathrm{Exp}\left(\frac{\Omega^2(t, t_0)}{2}\right) \end{aligned} \tag{G.18}$$

Thus

$$\begin{aligned} E(S) &= s_0 \mathrm{Exp}\left(\int_{t_0}^{t} w(u)du\right) E\left(\mathrm{Exp}\left(\int_{t_0}^{t} \xi(u)du\right)\right) \\ E\left(S^2\right) &= s_0^2 \mathrm{Exp}\left(2\int_{t_0}^{t} w(u)du\right) E\left(\mathrm{Exp}\left(\int_{t_0}^{t} 2\xi(u)du\right)\right) \\ &= s_0^2 \mathrm{Exp}\left(2\int_{t_0}^{t} w(u)du + 2\Omega^2(t, t_0)\right) \\ E(S^n) &= s_0^n \mathrm{Exp}\left(n\int_{t_0}^{t} w(u)du\right) E\left(\mathrm{Exp}\left(\int_{t_0}^{t} n\xi(u)du\right)\right) \\ &= s_0^n \mathrm{Exp}\left(n\int_{t_0}^{t} w(u)du + \frac{n^2}{2}\Omega^2(t, t_0)\right) \end{aligned} \tag{G.19}$$

Thus

$$\begin{aligned} \frac{\partial E(S)}{\partial t_0} &= s_0 \mathrm{Exp}\left(\int_{t_0}^{t} w(u)du + \frac{\Omega^2(t, t_0)}{2}\right)\left(-w(t_0) + \frac{D_2\Omega^2(t, t_0)}{2}\right) \\ \frac{\partial E(S)}{\partial s_0} &= \mathrm{Exp}\left(\int_{t_0}^{t} w(u)du + \frac{\Omega^2(t, t_0)}{2}\right), \frac{\partial^2(E(S))}{\partial s_0^2} = 0 \\ \frac{\partial E\left(S^2\right)}{\partial t_0} &= s_0^2 \mathrm{Exp}\left(2\int_{t_0}^{t} w(u)du + 2\Omega^2(t, t_0)\right)\left(-2w(t_0) + 2D_2\Omega^2(t, t_0)\right) \\ \frac{\partial E\left(S^2\right)}{\partial s_0} &= 2s_0 \mathrm{Exp}\left(2\int_{t_0}^{t} w(u)du + 2\Omega^2(t, t_0)\right), \\ \frac{\partial\left(E\left(S^2\right)\right)}{\partial s_0^2} &= 2\mathrm{Exp}\left(2\int_{t_0}^{t} w(u)du + 2\Omega^2(t, t_0)\right) \end{aligned} \tag{G.20}$$

$$\begin{aligned} \frac{\partial E(S^n)}{\partial t_0} &= s_0^n \mathrm{Exp}\left(n\int_{t_0}^{t} w(u)du + \frac{n^2}{2}\Omega^2(t, t_0)\right) \cdot \left(-nw(t_0) + \frac{n^2}{2}D_2\Omega^2(t, t_0)\right) \\ \frac{\partial E(S^n)}{\partial s_0} &= ns_0^{n-1} \mathrm{Exp}\left(n\int_{t_0}^{t} w(u)du + \frac{n^2}{2}\Omega^2(t, t_0)\right) \\ \frac{\partial^2 E(S^n)}{\partial {s_0}^2} &= n(n-1)s_0^{n-2} \mathrm{Exp}\left(n\int_{t_0}^{t} w(u)du + \frac{n^2}{2}\Omega^2(t, t_0)\right) \end{aligned} \tag{G.21}$$

Hence

$$\begin{aligned} &\frac{\partial E(S)}{\partial t_0} + \left(w(t)s_0 + \widetilde{\Gamma}(t, t_0)s_0\right)\frac{\partial E(S)}{\partial s_0} = s_0 \mathrm{Exp}\left(\int_{t_0}^{t} w(u)du + \frac{\Omega^2(t, t_0)}{2}\right) \\ &\left(-w(t_0) + \frac{D_2\Omega^2(t, t_0)}{2}\right) + \left(w(t)s_0 + \widetilde{\Gamma}(t, t_0)s_0\right) \\ &\mathrm{Exp}\left(\int_{t_0}^{t} w(u)du + \frac{\Omega^2(t, t_0)}{2}\right) = 0 \end{aligned} \tag{G.22}$$

$$\begin{aligned} &\frac{\partial E\left(S^2\right)}{\partial t_0} + \left(w(t)s_0 + \widetilde{\Gamma}(t, t_0)s_0\right)\frac{\partial E\left(S^2\right)}{\partial s_0} \\ &+ {s_0}^2\widetilde{\Gamma}(t, t_0)\frac{\partial^2\left(S^2\right)}{\partial {s_0}^2} = s_0^2 \mathrm{Exp}\left(2\int_{t_0}^{t} w(u)du + 2\Omega^2(t, t_0)\right) \\ &\left(-2w(t_0) + 2D_2\Omega^2(t, t_0)\right) \\ &+ 2s_0^2 \mathrm{Exp}\left(2\int_{t_0}^{t} w(u)du + 2\Omega^2(t, t_0)\right)\left(w(t) + \widetilde{\Gamma}(t, t_0)\right) \\ &+ 2s_0^2 \mathrm{Exp}\left(2\int_{t_0}^{t} w(u)du + 2\Omega^2(t, t_0)\right)\widetilde{\Gamma}(t, t_0) = 0 \end{aligned} \tag{G.23}$$

$$\frac{\partial E(S^n)}{\partial t_0} + \left(w(t)s_0 + \widetilde{\Gamma}(t,t_0)s_0\right)\frac{\partial E(S^n)}{\partial s_0} + {s_0}^2\widetilde{\Gamma}(t,t_0)\frac{\partial^2(S^n)}{\partial {s_0}^2} = s_0^n \mathrm{Exp}\left(n\int_{t_0}^t w(u)du + \frac{n^2}{2}\Omega^2(t,t_0)\right) \left(-nw(t_0) + \frac{n^2}{2}D_2\Omega^2(t,t_0)\right) + s_0^n \mathrm{Exp}\left(n\int_{t_0}^t w(u)du + \frac{n^2}{2}\Omega^2(t,t_0)\right)\left(w(t) + \widetilde{\Gamma}(t,t_0)\right)n + s_0^n \mathrm{Exp}\left(n\int_{t_0}^t w(u)du + \frac{n^2}{2}\Omega^2(t,t_0)\right)\widetilde{\Gamma}(t,t_0)n(n-1) = 0 \tag{G.24}$$

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## Conflicts of Interest

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